Accelerating Network Statistical Dynamics
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Abstract:
Recently there has been great interest in studying the complexity of networks in ever more complex detail that includes acceleration, a varying and possibly random strengths of links, the network topology and its hidden metrics. In short the complexification of network research and its application to ever more detailed physical dynamics has brought networks to the athermatics of statistical dynamics of interacting systems, of which the simpler models as these newly studied complex networks are special cases of, and which all can be described by a general field theory. Eschewing the full field theory however we present the intermediate theory a statistical dynamics of accelerating networks and of varying links weights, the current state of the art.

Introduction:
Recently there has been great interest in studying the complexity of networks in ever more complex detail that includes acceleration, a varying and possibly random strengths of links, the study of network topology and its hidden metrics. In short the detailed study or the complexification of network research and its application to ever more detailed physical dynamics has brought networks to the athermatics of statistical dynamics of interacting systems, of which the simpler models as these newly studied complex networks are special cases of, and which all can be described by a general field theory. Eschewing the full field theory for now however, we present the intermediate theory a statistical dynamics of both extensive and nonextensive statistical ensembles, of accelerating networks of varying links weights.

Great research efforts are underway to understand networks, these encouraged research directions resulting from varying successes of networks numerical models that though simple to the point of 'toy models' nonetheless have been capable of describing stylized facts of real world or physical networks and we include information networks such as internets as physical. The rules based simple computational simulations which emerge complex behavior even from sparse rules when analyzed have as mentioned shown the overall qualitative and in many cases quantitative behavior of networks of airports[5], internet, fungi[3], markets [7], and so forth.

Therefore some active researchers [5] have sought to study the topology and hidden metrics that underly the networks they utilize...small world networks a physically realistic model they focus on such models which are characterized by interactions that span the length scale of the system. Other researchers active in the field have chosen to focus on the observables and the statistics of the observables of networks which vary in time. These two approaches actually studying the same problem from two vantages. That is physical networks vary in time and accelerate and decelerate, have interactions on the scale of the system size, and have varying strengths of interactions between nodes...these then metric distances in information or geography that by definition have underlying generalized topological hidden metrics and dynamical evolution described by a dynamical Euler-
Lagrange mechanics though it be in an information space. We address the Euler-Lagrange dynamics of the hidden metrics in later work. For now we restrict our letter to describing the statistical dynamics of the observables of the time varying networks observables, following the current state of the art and the language that has been invented to describe the various non trivial phenomenon observed.

Introduction II

An accelerating network [3] is one which has the number of links related to the number of nodes as an equation of the form

\[ m(t) = p(N(t))^a \]

where the \( m(t) \) is the rate of change of the number of links and \( n(t) \) is the number of nodes is also described by its differential. The two constants are 'p' a prefactor from probability of occurrence and the \( \alpha \) is the acceleration where if >1 the network is accelerating and if zero it is static and if <1 it is decelerating.

Recently this description was considered too restrictive and generalization of this description obtained for the acceleration of a network

\[ \alpha(t) = \frac{d}{dt} \left( \frac{m(t)}{n(t)} \right) \]

where the acceleration constant is now time dependent and is defined by the rate of change in time of the node and link rates. The authors also state that velocity of the network is to be defined by

\[ v(t) = \frac{dM(t)}{dN(t)} = \frac{m(t)}{n(t)} \]

with other definitions of velocity and therefore acceleration as derivatives of an abstracted network coordinate as \( r(t)=N(t)M(t) \) possible though less amenable to interpretation.

We follow these definitions [3] (and references therein) in order to connect our results and this letter to the large body of research available both mathematical and numerical.

We mention that though the authors [3] begin their network analysis by positing discrete quantities for the network described by the characteristic parameter

\[ q(t) = \frac{2M(t)}{N(t)(N(t) - 1)} \]

and with discrete changes in time say, they subsequently approximate their derivatives sensibly by continuous quantities...We therefore omit the discrete preamble and begin our analysis from continuous variables as well, referencing the previous research and discrete derivation leading up to the definition of velocity and acceleration of networks as we have. We also mention that the issue of varying
weights of linkages is addressed by the cited authors as by a total weights time varying function
\[ L(t) = \sum_{i=1}^{M(t)} w_i \]
which also has its own velocity and acceleration paralleling the nodes and links velocities and accelerations. these then are the variables and therefore the main observables of the network system of [3].

Introduction III

The previous cited derivation addresses the issue of linkages' weights...These can be deterministic and or random in general. These can be in physical realizations geographical lengths and strengths (say frequencies or occurence of travel between the two nodes) such as in an airports network...that is links between two nodes such as Ohare Airport and Heathrow Airport have a weight characterized say by distance geographic and strength of interaction such as number of flights between the two nodes per some time scale...other factors can be additionally included as the components of what it means to describe a linkage 'weight'. We return to these issues later when we connect the concepts of networks' weights and information theoretical ineracting models of networks which we define here as a step further in complexification of descriptiveness towards the full field theory we alluded to.

One other point before proceeding to derivation...choice of statistics is crucial. The full field theory of which netwroks models, spin systems, game theoretic systems, cellular automata , rules based and multi agent models are special cases of are based on extensive composition of information and entropy, which as is well known derives exponential probability statistics and when variances are utilized (statistical 2nd moments) derive Gaussian Gibbs-Boltzmann type of statistical ensembles. Recently nonextensive composition of information measures of two systems or system events has been generalized to apriori nonlinear composition as \( I(AXB) = I(A) + I(B) + (q-1)I(A)I(B) \) with q a paremeter that when q->1 recovers the linear composition and extensive statistics of exponentials and Gibbs-Boltzmann. These apriori nonlinear statistics of nonextensitivity were discovered by Tsallis et. al. and we utilize them here for their high accuracy and simplicity of results when describing small world interactions..Note that a nonextensive field theory is a generalization of the extensive field theory and though we do not address these field theories formally it bears mentioning that as complexification of the theory of networks is made available by the research of authors active in the field the trend will be towards such unification. That is the hidden metric approach also seeks the Euler-Lagrange dynamics of the netwroks studied, which fluctuations enter into the Lagrangian Path integral formalism or its dual the many-particle Hailtonian theory and its Green's functions of Fokker-Planck and Schroedinger like parial differential equations....these the 'field theory's alluded to...the path integral formalism when generalized to nonextensive ensembles becomes modified to nonexponentials and rather power-law path integrals...Equivalently or dually one can approach the problem from information theory obtaining master equations and in the continuous approximation the Fokker-Planck equations of which Greens functions are the 'particle' duals of the fields theories. In order to close the circle we mention that the nonextensive statistics derive Fokker-Planck equations that are power-law in probability (or amplitudes) and whose Green's functions are also (as transition probabilities) of the power-law form rather than exponentials of Gaussians.
Derivations:

The accelerating networks are defined in the continuous approximation of the derivatives and that is our starting point.

We have then a number of nodes
\( \lambda(t) = \lambda(t) \),
and a number of links
\( \nu(t) = \nu(t) \),
at the current time that is dependent on the previous time's number of nodes and additionally and generally some nonlinear and random number of additional (or subtractive or zero) nodes as
\( \lambda(t) = \lambda'(t') + \eta(\lambda, \nu, t, \lambda', \nu', t') \)
where the number of nodes currently also depends on the numbers of links currently and as conditioned by the numbers of nodes and links in the past. The numbers of links similarly follow a general time dependent process that is nonlinear and random as
\( \nu(t) = \nu'(t') + \omega(\lambda, \nu, t, \lambda', \nu', t') \)
and where we define the random processes as indefinite integrals in time of nonlinear Wiener processes modified nonlinearly by their nonlinear diffusion coefficients here non constants...
\[
\eta(\lambda, \nu, t, \lambda', \nu', t') = \int [A_\lambda(\lambda, \nu, t, \lambda', \nu', t') + B_\lambda(\lambda, \nu, t, \lambda', \nu', t') \omega(t') dt', \\
\omega(\lambda, \nu, t, \lambda', \nu', t') = \int [A_\nu(\lambda, \nu, t, \lambda', \nu', t') + B_\nu(\lambda, \nu, t, \lambda', \nu', t') \omega(t') dt',
\]
This then when differentiated obtains the stochastic evolution of the number of nodes and the number of links
\[
d\lambda(t) = A_\lambda(\lambda, \nu, t, \lambda', \nu', t') dt + B_\lambda(\lambda, \nu, t, \lambda', \nu', t') d\nu(t), \\
d\nu(t) = A_\nu(\lambda, \nu, t, \lambda', \nu', t') dt + B_\nu(\lambda, \nu, t, \lambda', \nu', t') d\nu(t)
\]
where the random increments when differentiated are the unit variance Gaussian white noise Wiener process which is delta correlated and which is built upon by multiplication with the nonlinear diffusion coefficients the overall effect one of a nonlinear stochastic trajectory appropriate to modeling real world stochastic dynamics [1,2,4].

The stochastic microscopic processes (here Ito type) of trajectories are equivalent to the macroscopic evolution by the statistical distributions as by the vector Fokker-Planck equation (note we mix vector and subscript notation to remind the reader that the drift and diffusion coefficients differ for nodes and links)
\[
\nabla P(\lambda, \nu, t, \lambda', \nu', t') = -\nabla_\mu [A_\mu P] + \frac{1}{2} \nabla_\mu^2 [B_\mu P].
\]
This Fokker-Planck PDE partial differential equation has known short time transition probability solutions...For long times solutions have also been made available case by case as the solution depends either on the type of PDF transformation from unknown nonlinear Fokker-Planck P(x) say to a known solution P(y) of a diffusion equation is as P(x)dx=P(y)dy at the macro level of PDEs, and as Ito's formula of variable transformations D[f,g]=D[f]g+D[g].f+D[f]D[g]...Therefore transformations to a linear stochastic process can be made and solutions obtained...we refer the interested reader to the excellent books by H. Risken and C.W. Gardiner and our referenced papers and references therein. To summarize solutions are possible for these equations.

That is an SDE of the form
\[ dx = a(x,t)\, dt + b(x,t)\, dW(t) \]
can be transformed to another variable as
\[ dy = \frac{\partial y}{\partial t}\, dt + \frac{\partial y}{\partial x}\, dx + \frac{1}{2}\left( \frac{\partial^2 y}{\partial x^2} \right)\, dx^2 \]
where the new variable is time independent explicitly eliminating the first term, and the replacement or substitution of the dx SDE into the dy SDE obtains
\[ dy = \left[ \frac{\partial y}{\partial x}\, a(x,t) + \frac{1}{2}\left( \frac{\partial^2 y}{\partial x^2} \right)\, b^2(x,t) \right] dt + \frac{\partial y}{\partial x}\, b(x,t)\, dW(t) \]
and with choice of defining how this new SDE has its drift-diffusion coefficients partitioned!

For example making the diffusion coefficient a constant
\[ \frac{\partial y}{\partial x}\, b(x,t) = D_y \]
 obtains the variable transformation
\[ y(x,t) = D_y \int_0^t b^{-1}(x'',t)\, dx'' \]
for the SDE
\[ dy = C(y,t)\, dt + D_y\, dW_y(t) \]
which is solved by the Fokker-Planck equation (recall the vector Fokker-Planck is separable into its vector components and presents no great extra difficulty of solution)
\[ \nabla_y p(y,t) = -\nabla[C(y,t)\, p] + \frac{D_y^2}{2} \nabla^2[p(y,t)] \]
which has the short time transition probability solution
\[ p(y,t | y',t') = \frac{1}{Z} e^{\frac{-(y-y')^2-C(y',t')\, k^2}{2D_y^2(t-t')}} \]
Alternatively insisting that the drift coefficient is zero valued (drift-less) and with the substitution of this equation into the diffusion coefficient obtains a variable transformation in terms of the drift coefficient obtaining
\[ y(x,t) = 2D_y \int_0^x \left( \int_0^x \frac{a(x'',t)}{b'(x'',t)}\, dx'' \right) dx'' \]
for the resulting 'simple' SDE
\[ dy = D_y dW_y(t) \]
for which the Fokker-Planck equation is reduced to the diffusion equation
\[ \nabla_t p(y,t) = \frac{D_y^2}{2} \nabla_y^2 [p(y,t)] \]
and for which drift-less constant diffusion coefficient Fokker-Planck is the diffusion equation solution
\[ p(y,t) = \frac{1}{Z} e^{-\frac{y^2}{2D_y^2t}}. \]

We also note that the Fokker-Planck equation is the asymptotic limit of the master equation of the system. That is the discrete derivation for generally random evolution should also obtain some transitioning discrete random process governed by the master equation which in the continuous approximation can be shown to be the generally nonlinear Fokker-Planck equation. As these not-so-well-known relationships have been discussed at length elsewhere[4] we do not repeat the derivations but reference again the books of Gardiner and Risken and the references cited.

Derivations II:
We have then the Fokker-Planck PDEs and equivalently the SDEs stochastic differential equations of the accelerating networks processes of nonlinear statistical dynamical evolution...

The processes assumed nonlinear generally and separate into their vector components though obtaining coupled SDEs, we mention the apriori nonlinear statistics and their applications to these nonlinear SDEs and Fokker-Planck evolutions of the accelerating networks.

The nonlinear Itô type of SDE
\[ dx = a(x,t)dt + b(x,t)dW(t) \]
is mapped to the special case of the Tsallis nonextensive statistics with the simple replacement of the nonlinear diffusion coefficient as
\[ b(x,t) = \sqrt{DP(x,t)^{1-q}}. \]
Ignoring the drift coefficient for now the Fokker-Planck equation obtained equivalently (see Gardiner[4]) [1,2] is
\[ \nabla_t P(x,t) = \frac{D}{2} \nabla [P^{2-q}] \]
where the power-law distribution is
\[ P(x,t) = \frac{1}{Z}[1 + \beta(t)(q-1)(x^{-} < x^{+})^{2}]^{\left(\frac{1}{1-q}\right)} \]

and where we have set the simple drift coefficient to zero earlier, we mention another special innovation of this statistics...that is whether with linear drift or more complicated yet symmetrical drift coefficients, increasing measure of the sequential (here) nonlinearity by the tunable parameter \( q \) allows one to describe different nonlinear processes by the same form of power-law distribution but with different \( q < q' < q'' < \ldots \) values. For example the two SDEs
\[
\begin{align*}
    dx &= \sqrt{D_x} P(x,t)^{(1-q)} dW_x(t) \\
    dy &= y^2 dt + \sqrt{D_y} P(y,t)^{(1-q)} dW_y(t)
\end{align*}
\]
actually describe the same stochastic nonlinear process along different parametrizations of the sample space of trajectories here parameterized by \( q < q' \). Continuing this to generally nonlinear drift coefficients obtains the same process as
\[
\begin{align*}
    dz &= a(z,t) dt + \sqrt{D_z} P(z,t)^{(1-q')} dW_z(t)
\end{align*}
\]
where \( q'' < q' < q \) and so on until in fact the full nonlinear drift coefficient is the actual deterministic dynamics for which the \( q \) value is the extensive \( q > 1 \) value obtaining a constant valued diffusion 'coefficient'. Note the analogy with the previous extensive statistics and stochastics' Ito formula transformation between one SDE to its equivalent alternative trajectory parameterization SDE... here the previous extensive 'nonlinear variable Ito transformation' is accomplished by the simple expediency of the 'transformation' of the nonlinear parameter itself though can be made precise and formal ([1,7] and references)...

These brief details merely serve as a preliminary to the application of the stochastic calculus to the problem at hand.

Derivation III:

The problem of generally nonlinear evolution of an accelerating network is then one that we have shown at least for the continuous approximation, can be a) solved exactly... b) can be mapped to nonextensive apriori nonlinear statistics and solved exactly.

That is the previous model we generally derived for nodes and links as
\[
\begin{align*}
    d\lambda(t) &= A_{\lambda}(\lambda,\nu,t | \lambda',\nu',t') dt + B_{\lambda}(\lambda,\nu,t | \lambda',\nu',t') dW_{\lambda}(t) \\
    d\nu(t) &= A_{\nu}(\lambda,\nu,t | \lambda',\nu',t') dt + B_{\nu}(\lambda,\nu,t | \lambda',\nu',t') dW_{\nu}(t)
\end{align*}
\]
is simply re written as
\[
\begin{align*}
    d\lambda(t) &= \sqrt{D_{\lambda}} P(\lambda,\nu,t)^{(1-q)} dW_{\lambda}(t) \\
    d\nu(t) &= \sqrt{D_{\nu}} P(\lambda,\nu,t)^{(1-q)} dW_{\nu}(t)
\end{align*}
\]
which is a history following and dependent nonlinear stochastic vector evolution solved immediately as discussed from its Fokker-Planck PDF (called the Tsallis-Zanette equation) equation as

\[ P(x,t) = \frac{1}{Z}[1 + \beta(t)(q-1)(x - <x>)^2]^{1-\eta} \]

where the vector is comprised of the components of the nodes and links space 'rates' coordinates as

\[ \vec{x} = \vec{\lambda} + \vec{\lambda} + \vec{v} + \ldots \]

We leave the mean as a reminder that linear drift is easily accommodated into the equation however here it is set to zero.

Discussion :

We have the model of history following processes SDEs of the nodes and links variables observables of the accelerating network, and generally the nonlinear results of the PDFs and as well as the specialized nonlinear Tsallis-Zanette result. The TS result particularly allows us to immediately discuss the variance of the velocity which we obtain as

\[ V(t) = \frac{dM(t)}{dN(t)} = \sqrt{\frac{D_v P^{1-q}}{D_\lambda P^{1-q}}} \frac{dW_v(t)}{dW_\lambda(t)} \]

from the TS result of the SDEs the stochastic evolution of the nodes and links equations. Upon cancellation of the q dependent probabilities we are left with simple Gaussian white noise, and we replace the random variables by their variances (the means are zero) of unit value for the nodes and links' driving white noise, and then take the square root to show that the variance of the velocity is a constant implying the velocity itself is within a constant +/- standard deviation.

\[ <V^2>(t) = \frac{D_v}{D_\lambda} \]

Alternatively we could perform numerical simulations to show this result from computational perspectives and we perform this later. We also note that such a constant velocity result is perhaps 'confirmed' or rather exhibited by the data analysis performed by N.F. Johnson et. al. [3] on the Wikepedia English German and Japanese networks, and where the velocity is the slope of the graph of the total number of links to the total number of nodes and is nearly a constant throughout several orders of magnitudes.

Alternatively to the TS result, we should note that more complicated analysis would be required if we utilized the generally nonlinear form equivalent in the sample space to the nonextensive SDEs. The form of the velocity again of SDES of the form

\[ dx = a(x,t)dt + bdw(t) \]

\[ dy = A(y,t)dt + BdW(t) \]
with here the constant diffusion coefficients (that is the intermediate nonlinear form after transformation to the drift frame) such that one obtains the result that the variance of the velocity is similarly a constant of the ratio of the diffusion constants this after careful application of Ito's stochastic sequential differentiation rule and formula.

Discussion II:
Hidden Metrics are sought for [5] in order to extract the dynamics that supposedly direct a network. Recently pointed out that this is not as complicated a problem as it seems. One merely defines the network whether geographical or information abstracted and then one obtains the effective observables' (here in [3]'s model one of links and nodes and their velocities and accelerations) master equation and its asymptotic and approximate continuous limit...Then one transforms to the driftless frame such that the drift-diffusion is repartitioned solely with the diffusion coefficient and then utilizes well known results (Gardiner and Risken [4]) that the hidden metric of topology is related to the inverse of the diffusion coefficient as

\[ g^{ij} = D_{ij} \]

this notation indicating inverse component by vector component. For the (drift-less) diffusion frame generally nonlinear PDEs derived in previous sections this tells us that the hidden metric is also a function of the drift and diffusion coefficients of the full nonlinear general process. Note that the full generally nonlinear process' drift and diffusion coefficients are obtained simply enough by the Peinke-Renner probabilistic(see references [1,2]) method...that is the SDEs of the form

\[
\begin{align*}
\frac{dx}{dt} &= a(x, y, t)dt + b(x, y, t)dw(t) \\
\frac{dy}{dt} &= A(x, y, t)dt + B(x, y, t)dW(t)
\end{align*}
\]

are easily determined from real data. We omit the explicit and tedious writing out of the inverses of diffusion coefficients of the transformed SDEs (to the driftless frame or sample space) either in the general nonlinear case or the TS specialized case, reminding the reader that the hidden metrics of topology are of the Euclidean or Minkowskian (depending on application) form

\[ dS^2 = f(x, y, t)dx^2 + g(x, y, t)dy^2. \]

Discussion III:
However this is not fully the sought for hidden metric spaces result...researchers want to connect the abstract information space of a network with the topological 'hidden' metric [5]and in order to obtain the Euler-Lagrange 'hidden' dynamics of the networks... The leap from topological metric to Euler-Lagrange dynamics is well known (it yields for example Newton's laws F=ma, or Einstein's generalization of Newton's laws or in terms of field theory the generalized field equations etc...) and will not be reviewed here.
However the concept of how to perform this hidden metric extraction for the other research perspective [5] seeking to describe abstract information 'small world' networks is similar to the accelerating networks models of [3]. In order to describe abstract information networks (or hybrid ones) they must posit an information measure between nodes, a Hamming distance itself a metric by the way...This puts the model into an abstract geometrical space where numerical distances (vectors) exist between nodes...immediately the results here apply that these nodes and their topological links are themselves characterizing observables of the network. The information Hamming distance for example between the nodes then becomes a weight in this perspective! More generally the information node-node measured by the Hamming distance between nodes is a metric measure of information, and its summation in discrete form when divided by the total number of links' information content is merely the expectation value (sum integral, average) of the Discrete Shannon information measure (or its partial summation) expressed in one particular way. Others are available...as we approximated derivatives in continuous variables and moved to probabilistic models of the evolution of the network, we can generalize the time dependent total weights function from summation to an integral of the known form

\[ <I(t)> = c \int P(\lambda, \nu, t) \ln P(\lambda, \nu, t) d\lambda d\nu \]

this the C. Shannon extensive (logarithms and exponentials) continuous form, its nonextensive C. Tsallis form (with same variables)

\[ <I(t)> = c \int \frac{P - P^q}{(1 - q)} d\lambda d\nu \]

is discussed elsewhere. We leave the integration indefinite here as issues of partial summation and therefore integration limits and inclusion of other degrees of freedom from information Hamming distances to discrete multi-state vectors are a case by case or model specific artifact. That is we can utilize the vector consisting of the nodes and links and furthermore the additional degrees of freedom whether discrete or continuous.

To summarize, information networks can be mapped to information metrics imparting abstract topology AND information measures that are Shannon-like and though they be possibly escort probability like partial sets of the overall network informational content. We return to these concepts in later work.

The information geometry abstracted as discussed, one has a space-like topology of the information network of n-Dimensions, this actually depending on how one describes the degrees of freedom available...THEN one can obtain the n-Dimensional Fokker-Planck of the network similarly to how we obtained the Fokker-Planck PDEs of the node-link velocity-acceleration model and then transform to the diffusion frame and 'read off' the not-so-hidden metric coefficients.
Derivation IV:

We derived our model by adding generally nonlinear random functions to the evolution of the number of nodes and links...In stochastic systems this is called the 'ad hoc' method. It can be made more formal and we briefly discuss how in this derivation section.

We have the E.T. Jaynes theory of maximum entropy, and its equivalent the C. Shannon information theoretic method as starting points...We can add by Legendre transform any observable we wish to our model. We choose to add the observation of the statistical moments of the randomly varying in time quantities of nodes numbers and numbers of links as

\[ \frac{d}{dt} \langle I(t) \rangle - d[\alpha(t)(\langle \lambda - \langle \lambda \rangle \rangle^2 + \langle v - \langle v \rangle \rangle^2)] = 0 \]

and we vary this Legendre transform weighted by Lagrange multipliers and depending on whether we utilized the logarithmic information measure of C. Shannon or the power-law information measure of C. Tsallis we obtain the Gaussian statistics of the Fokker-Planck of generally nonlinear form or the Tsallis-Zanette equation and for the Fokker-Planck with q-parametrized powers of the PDF. The remaining connection is straightforward as it is merely the equivalence between a Fokker-Planck PDE and its microscopic trajectory SDE stochastic differential equation. discussed at length by Gardiner and Risken [4] and other authors. Therefore for completeness we consider that we have derived the model from microscopic stochastics and from macroscopic statistics obtained from the state functions such as entropy and information.

We should mention that we can add weights (the [3] form) as statistical moments in the same way, this adding one or more extra degree of freedom 'dimension' to our description...We can partition the weights and describe discrete-continuous processes such as the weights consisting of both information Hamming distances continuously enumerated, and with discrete two or more states of each node and the interaction between nodes or the weight having a discrete component. For example in a social network we may describe information metrics of content in the weight and add discrete valued information of each nodes' (person or additionally cluster size as clique) receptivity to trend following such that each node is additionally embued with three valued bias such that the ith node has three valuedness such as trend following, neutral and trend averse choices

\[ j_i = \{+1, 0, -1\} \]

and where the interaction between nodes is also chosen from a probability distribution straightforwardly and on the same footing as our derivation, as the discrete valued degree of freedom is merely a 'spinor' or 'tensor' complexification to the previous derivation...to illustrate

\[ < I(t) > = c. \sum_{\sigma=1}^{3} \int P_\sigma(\lambda, v, t) \ln P_\sigma(\lambda, v, t) d\lambda d\nu \]

the information measure becomes a 3X3 matrix (of a tensor ), the interaction between node-node is included in some static or dynamic fashion such that (we focus on the matrix complexification here)

\[ d < I(t) > + d[\alpha(t)(\langle \lambda - \langle \lambda \rangle \rangle^2 + \langle v - \langle v \rangle \rangle^2)] - I < c_\sigma > \] = 0

The reader should recognize a spin-spin model here.
In fact this is one of the main points we are making...As one complexifies the modeling of the network beyond the simple rules based iteration network...that is as one adds weights, selectivity or bias, information content, 'strength' of interaction... one discovers that one is heading for the general field theory realm...with the example here a simple a super-spin-like model we utilize regularly in modeling interacting networks of traders in financial markets[6,7].

From another way of stating this, the network research having began from iterative rules based computation approaches decades ago, and from simple models of observed interactions between 'nodes' that were considered disparate as whether they were human or electrical generators or airports or plants or fungi, has been quite literally immersed in the forest and seeing only trees...with the past ~ 15-20 years of complexity research however, there has gradually arisen an awareness that the researcher he or she is catching glimpses of the the whole network forest despite the trees in the way...This is to be contrasted with the need by scientists in the past century to describe many-particle systems of Avogadro's number in scales where their only recourse in trying to describe the average tree was to use forest descriptive tools from the outset, that is methods such as field theoretic methods, many-particle methods (its dual), and other statistical methods dependent on large 'numbers' (perhaps enumerations or possibilities is more accurate) in scale. Recall also that shortly after J. Willard Gibbs and the Gibbs-Boltzmann statistics Einstein came along and described both Brownian motion and quantum statistical mechanics of photons emitted from a cavity by a convolution of integrals weighted by statistical distributions. We mention that the derivations by Einstein if generalized to transition probabilities are the Green's functions of many-particle physics and if by Lagrange dynamics the Path integrals of field theory. Also we note that for illustration we prefer the information theoretic and equivalently the maximum entropy method derivation due to its almost informal simplicity in obtaining the results we are seeking. The connection to Hamiltonian dynamics and Lagrangian dynamics is straightforward as we have discussed.

Numerical Results And Modeling Simulations:

We simulate the theory obtaining numerical results. The theory we refer to is the nonextensive q>1 theory, as it is the more robust theory, and we can always recover the q->1 theory from it, though this may 'miss' the mark in accuracy and in modeling the full nonlinear behavior of complex networks.

We are interested in the full nonextensive theory reproducing the velocity of M links and N nodes and therefore its acceleration. We model then the full nonextensive stochastic evolution or trajectories of the links and nodes.

These are coupled by our theory as the nonextensive PDF is a 2Dimensional vector of links and nodes. We can simulate this directly, or perform a simple integration to obtain separately the PDFs of links and the PDFs of nodes. These are simulated as stochastic trajectories, with 1000 data points each, and over 500 runs for both simulations. This in turn is utilized to obtain 500 runs of velocities and this in turn is averaged. The result is that we obtain a graph of the average velocity that reproduces the graph of the velocity as the number of links divided by the number of nodes reported by N.F. Johnson et. al. for accelerating networks and for the Wiki data.
Fig. 1. The graph of the nonextensive q-parameterized joint PDF of \( m = \text{links} \) \( n = \text{nodes} \) with \( q \sim 1.4 \) with \( 1 < q < 5/3 \), inverse variance 30, PDF normalized to unity.

Fig. 2. The graph of the integrated PDF of \( m = \text{links} \) blue. The integration introduces a renormalized partition function. The nonextensivity parameter is \( q \sim 1.4 \) and the sharp peak and slow decay of large values outliers is seen clearly compared to the same variance graph of the extensive \( q > 1 \) Gaussian.
Fig. 3. Stochastic trajectories of n=nodes. The nonextensive parameter is $q \sim 1.4$, inverse variance beta is 30, drift is 0.01 and the integrated PDF is renormalized to unity. There are 1000 data points per normalized to unity run, and there are 500 runs per simulation.

Fig. 4. Stochastic trajectories of m=links. The nonextensive parameter is $q \sim 1.4$, inverse variance beta is 30, drift is 0.05 and the integrated PDF is renormalized to unity. There are 1000 data points per normalized to unity run, and there are 500 runs per simulation.
Fig. 5. The averaged runs of the Fig. 3. and Fig. 4. simulations respectively n=nodes and m=links.
Fig. 7. The graph of the averaged velocity...q~1.4, and 500 runs of 1000 data points. The linear like rise to run of the average of the velocity (NOT the velocity of the averages) for inverse variance 30.

Fig. 8. The graph of the averaged velocity...q~1.4, and 500 runs of 1000 data points. The linear like rise to run of the average of the velocity (NOT the velocity of the averages) for inverse variance 50, noting the 'tighter' fluctuations about the line as variance decreases.
Discussion And Conclusion IV:

So the complexification or generalization to a more robust model capable of describing accurately both detailed additional degrees of freedom as well as the nonlinearity introduced into the variables' stochastic evolution, these as additional components of the discrete-continuous vector is easily enough accomplished for accelerating networks, these time dependent networks that expand and or contract, and for which topology hidden metrics are obtained (nontrivially for the weighted linkages and structured accelerating networks) from the macroscopic evolution equations of the probability density functions as discussed. The extraction of the hidden metric of small world networks is as a followup easily enough accomplished as well, however we leave application to specific small world networks to future work.

The continuous derivatives approximation (the diffusion approximation follows) taken, the generalization from master equation to PDEs of Fokker-Planck form are easily enough accomplished.

The results by N.F. Johnson et. al. for accelerating networks (not shown, see references) are here reproduced, and from the nonextensive and nonlinear generalized theories. The results summarized are that real data of Wikipedia shows such a linear nodes to links graph, accelerating networks exhibit periods of acceleration followed by near steady state conditions when initially formed, however these are a 'slice' of the bigger picture...networks evolve from their formation and grow, then they undergo periods of expansion and contraction as well as steady state or near constant evolution and as complex interacting nonlinear dynamics these are most likely self-similar across all scales. The results of our simulations, showing how accelerating networks dynamically evolve are a description of only one facet, the other facets if they were considered would be similalry described by the information theory we discussed, however the bigger or overall life cycle of a network surely having additional degrees of influence that are yet to be determined, these consisting of external perturbations and influences as well as the inherent dynamical features of the network and the interaction of these. Perhaps the research into topology of hidden metrics will shed further light on this 'big picture' evolution.

In the mean time the current letter discusses the random evolution of a network and generalizes the mathematical tools to the nonlinear composition law of information \( I(AXB) = I(A) + I(B) + (q-1)I(A)I(B) \) and therefore to a generalization of Shannon's information theory and E.T.Jaynes maximum entropy theory and the resulting statistical framework accurate for nonlinear stochastic evolution that results.

We obtain by simulation of hundreds of runs the result that the graph of the number of nodes graphed against the number of links if averaged does indeed obtain a linear relationship and which is sensitive to variance or fluctuations in spread around the clearly visible linear relation...in stochastics 'lingo' growth of each stochastic variable by different linear amounts will obtain such averaged linear relationships as averaged velocity and as the averaged ratio of the growth...Other features such as the weighted links are not addressed directly here.. We note however that weighted links introduce further fluctuation
overall to the number of links, this causing further deviation from uncorrelated random or stochastic trajectories, this the regime of nonlinear correlations described well by the apriori nonlinear statistics and power-law PDFs. In fact we simulate intentionally a high degree of nonlinearity $q \approx 1.4$, where the deviation from uncorrelated evolution is high, and is discussed in the figures as we compare the two evolutions...The weights would introduce further fluctuations, meaning clustering of small fluctuations and statistically significant occurrences of large changes or jumps in values...some links would be infrequent, some links would be very large comparatively, this exhibited for example by airport networks where there exist hubs of frequent large valued linkages, and airports of infrequent utilization comparatively...This would also occur in internets networks, as certain topics and knowledge or information areas are frequently visited creating 'hubs' of high utilization, whilst other informational or knowledge areas would not see the same level of inquiry. This again translating to high frequency of small valued variables and yet the occurrence of outliers infrequent yet statistically significant to the network events. Note that this merely describes the statistics accurately...It does not describe the external perturbations and or influences and their interaction with the inherent dynamics as such influences and interplays would contract or expand a network, which I suspect we will have to add as features of the external environment the network interacts with,... modeling such influences on growth and on limits of growth and contraction could be very similar to the modeling of a financial market network which must take into account risk aversion and interest rates and other 'economics' factors to model the expansion or contraction of a market comprised of an interacting network of traders and financial agents. Biological networks similarly influenced and or limited in expansion and contraction by external factors such as food supply, environmental utilization and health, predator-prey relations and so forth.

However often it is the accuracy of statistics in describing the nonlinear 'high frequency' evolution that is the problem that has not been described precisely, or has not been amenable to description and which stands in the way of further progress in understanding a nonlinear complex system... As such we have we feel provided the generalization to such accurate statistics and stochastic dynamics and towards future progress.

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