The Dakotas War

(A Study of Static Defenses as an Application of Geometry)

by Victor Aguilar, author of *Geometry–Do*

The U.S. Federal Government, in its infinite wisdom[1], has imposed a heavy tax on Peppermint Schnapps in their ongoing effort to promote sobriety (Rudy Giuliani notwithstanding), as they have noticed a tendency of hard-working Dakota farmers (in the summer) to spend their winters hunched over a fishing line stuck through a hole in lake ice inside a tiny and rather chilly hut taking shot after shot of their favorite beverage.

"We're from the government – we're here to help!" the bureaucrats loudly announce.

Naturally, the first response considered by the Dakotans is total war. The Governor of North Dakota, Charlie Brown, and the Governess of South Dakota, Peppermint Patty, ~~banged~~ put their heads together and came up with a plan. They would succeed from the union, declare unconditional war on those rascally Americans and, simultaneously, the even more rascally Canadians to their north[2], and fight to the death for their right to drink Peppermint Schnapps – in volume! Central to her plan (both literally and figuratively) is a nuclear power plant from which Gov. Patty can obtain the plutonium needed to build – *drum roll, please!!!* – the Dakotas Bomb!

I cannot advise them on their political calculations, but I will help them with tactics. General Lucy van Pelt observes that Rapid City, Sioux Falls, Fargo and Belfield (famous for its *Superpumper™* gas station[3]) form a rectangle, $\overline{EFGH}$. Lucy intends to construct a triangle of military bases (with rapid-response troops and anti-aircraft guns that can also enfilade the highways) at Rapid City and on highways 29 and 94; that is, they will find J on $\overleftrightarrow{FG}$ and K on $\overleftrightarrow{GH}$ such that $\overline{EJK}$ is an equilateral triangle with the nuclear power plant at its center, equidistant from each base.

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[1] *Yeah right!!!*

[2] Yes, there actually is something to the north of the Dakotas.

[3] Paul Yiu plugged a commercial product – *Geometer's Sketchpad™* – in the abstract of a paper about elegance in mathematics, so why not mention *Superpumper™* here? www.forumgeom.fau.edu/FG2005volume5/FG200512index.html

Paul Yiu has written extensively – a whole four lines – on this geometric construction. Like a lot of mathematicians, he overuses the letters A, B, C, D, so I changed his notation to ours (p. 149)[4]:

> *This construction did not come from a lucky insight. It was found by an analysis! Let $\overline{EF} = \overline{GH} = a, \overline{FG} = \overline{EH} = b$. If $\overline{FJ} = y, \overline{HK} = x$ and $\overline{EJK}$ is equilateral, then a calculation shows that $x = 2a - \sqrt{3}b$ and $y = 2b - \sqrt{3}a$. From these expressions of x and y, the above construction was devised.*

Isn't that amazing? Paul Yiu "devised" a construction based entirely on two algebra equations. Because of the "strength and power" of algebraic calculations, we can be rid of all geometry theorems and replace the entire geometric proof with a couple of algebra equations. *Woo hoo!*

> *D. E. Smith[5] (p. 95) explained that the teaching of constructions using ruler and compass serves several purposes: "it excites [students'] interest, it guards against slovenly figures that so often lead them to erroneous conclusions, it has genuine value for the future artisan, and it shows that geometry is something besides mere theory…" For all the strength and power of algebraic analysis, it is often impractical to carry out detailed constructions with paper and pencil, so much so that in many cases one is forced to settle for mere constructability… We focus on incorporating simple algebraic expressions into actual constructions using the Geometer's Sketchpad™.*

After quoting Smith (co-author of Wentworth's *Plane Geometry*) on the importance of geometric constructions using ruler and compass, Paul Yiu slides into extolling the "strength and power" of algebraic analysis and dismisses all geometric constructions as impractical, settling for "mere constructability," which his amazing four-line "proof" apparently represents. Then he smoothly transitions into his real job, which is selling *Geometer's Sketchpad™* for *McGraw-Hill*. Amazing. In his next performance, Paul Yiu will demonstrate how fast he can pedal a unicycle backwards!

Now let's do it right! Consult *Geometry–Do* regarding any theorems you are unfamiliar with. Specifically, you will need to know about the centerline theorem (p. 18), the transversal theorem corollary (p. 91), the triangle frustum mid-segment theorem converse (p. 105), SAS (p. 15; everybody knows that one, even *Common Core* fools), and the medial triangle theorem (p. 103).

Lemma

Let ρ be a right angle, σ be a straight angle and φ be the interior angle of an equilateral triangle. φ trisects σ and $\frac{1}{2}\varphi$ trisects ρ. The exterior angle of an equilateral triangle is $\rho + \frac{1}{2}\varphi$.

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[4] www.researchgate.net/profile/Victor_Aguilar4/publication/291333791_Volume_One_Geometry_without_Multiplication

[5] Smith, David Eugene. [1911] 2013. ***The Teaching of Geometry***. Los Angeles, CA: HardPress Publishing

Dakota Defense Problem

Given a rectangle, $\overline{EFGH}$, find J on $\overleftrightarrow{FG}$ and K on $\overleftrightarrow{GH}$ such that $\overline{EJK}$ is an equilateral triangle.

Solution

Build an equilateral triangle on $\overline{GH}$ with its apex, M, on the same side of $\overleftrightarrow{GH}$ as E and F. Let $J = \overrightarrow{EM} \cap \overrightarrow{GF}$. Build an equilateral triangle on $\overline{FG}$ with its apex, N, on the same side of $\overleftrightarrow{FG}$ as E and F. Let $K = \overrightarrow{EN} \cap \overrightarrow{HG}$. It is proven below that $\overline{EJK}$ is equilateral.

Proof

By the center line theorem, M is on the mediator of $\overline{GH}$ and so, by the transversal theorem corollary and the triangle frustum mid-segment theorem converse, M is the midpoint of $\overline{EJ}$. Analogously, N is the midpoint of $\overline{EK}$. $\overline{EF} = \overline{HG} = \overline{MG}$; and, by the lemma, $\angle EFN = \angle MGN$; also, $\overline{FN} = \overline{GN}$. Thus, by SAS, $\overline{EFN} \cong \overline{MGN}$, which holds the equalities $\angle FNE = \angle GNM$ and $\overline{NE} = \overline{NM}$. If the angle between these sides, $\angle MNE$, equals φ, then $\overline{MNE}$ is equilateral. $\angle MNE = \angle GNF + \angle FNE - \angle GNM = \angle GNF = \varphi$. By medial triangle theorem I, $\overline{MNE}$ equilateral implies that $\overline{EJK}$ is equilateral. ■

This is orange belt; proof that it works for a parallelogram is black belt and is left as an exercise.

When Paul Yiu states this problem, he claims that the equilateral triangle is "inside the rectangle," which is clearly not always true. The triangle is inside a square and it is inside a few rectangles that are almost square (within two by root three; $2 \times \sqrt{3}$); but, contra Paul Yiu, "inside" is not generally true. But Paul Yiu just found this problem in somebody else's textbook, and, by happenstance, they had drawn it with the triangle inside the rectangle. It is easy to leap to conclusions about results that you are stealing!

1. Paul Yiu claims to have "devised" the solution to this problem entirely with two algebra equations. This is not true. He just smeared some algebra on top of someone else's work.

2. Paul Yiu claims that his solution always inscribes the equilateral triangle inside the rectangle. This is not true. The triangle is only inside rectangles that are almost square.

These are the types of mistakes that happen when one smears some algebra on top of a geometry theorem that one just finds on the internet. Paul Yiu found the algebraic lengths of a couple of segments; he did not prove anything. The person he was stealing from happened to draw a figure with the triangle inside the rectangle, so Yiu leaped to the conclusion that it always is. There is a reason why geometers prove theorems; it is so we are sure that we know what we are doing.

There are a lot of people in America who are afraid of geometry and their cowardice in the face of a subject that they do not understand drives them to attempt to replace geometry with algebra. To save geometry, we must shame these people by demonstrating that they do not understand geometry *or* algebra. American high schools will soon abandon geometry in favor of – *God forbid!* – statistics; it needs to be saved from Paul Yiu, Agostino Prástaro, *et. al.*

The Dakota Defense is of interest to military cadets because soldiers are sometimes tasked with building something that they know will be targeted by the enemy – say, a munitions dump – in the middle of open farmland that paved roads have cut into rectangles. They know:

1. Their bases must be on paved roads so they can quickly move to confront enemy infantry approaching from anywhere, and so they can enfilade the roads to hit enemy vehicles.

2. Enemy aircraft are best met by anti-aircraft guns at the vertices of an equilateral triangle.

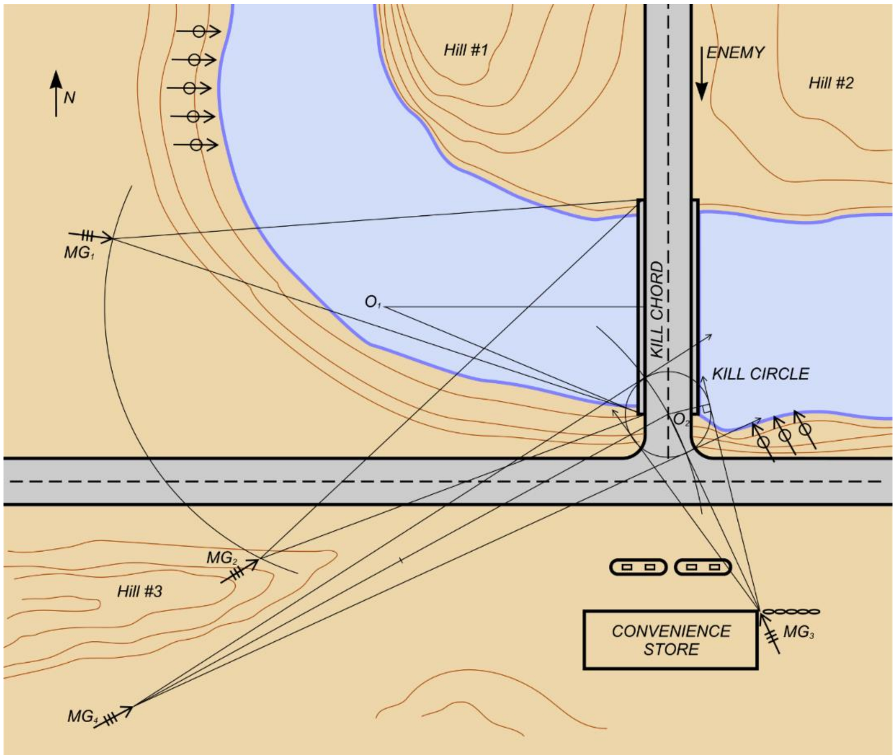
Helping the U.S. military fight more effectively is also why my textbook, *Geometry–Do*, emphasizes machine gun emplacement. Russia teaches real geometry in their high schools, not that bogus *Common Core* drivel, and we must too if we are going to fight them. It is ridiculous that American military officers go into battle without a scientific approach to laying ambushes.

Thus, let us now consider how geometry addresses machine gun emplacement: Top traverse is an angle, the lateral limit of a gun's ability to traverse. A gun's field of fire are all the points inside this angle; one may also say that these points are visible under the angle. If this angle is interior to a triangle, the kill chord is the opposite side. Bunkers with identical slits are positioned on an arc whose center is on the mediator of the kill chord such that the angle subtended at the center by the kill chord is twice that of the gunners' top traverse as defined by the width of their slits.

We will define the kill circle of two guns to be the circle of largest radius in which every interior point is within the field of fire of both guns. If two guns are not equally distant from the center of their kill circle, then, to make the quadrilateral over which their fields of fire overlap tangential, the far gunner must be instructed to traverse over an angle less than his top traverse.

The near gunner defines the kill circle. Draw a ray from his position through the kill circle center and then draw rays on either side to define his top traverse. Drop perpendiculars from the kill circle center to these rays to find the touching points and then draw in the kill circle.

The far gunner must reduce his traverse. Draw a ray from his position through the kill circle center, bisect it and then draw an arc around its midpoint through the kill circle center. Where it intersects the kill circle are the touching points of rays that define his new top traverse.



Scenario #1: The enemy comes in two BTR-80 armored personnel carriers, each armed with a 14.5 mm auto-cannon and coaxial 7.62 mm machine gun. They hope to secure the 100-meter bridge and seize the 10,000 liters of diesel at the store. You stand in their path with four M2 machine guns (top traverse: $22.5°$) and two dozen LAW rockets (effective range: 200 meters).

The kill chord of MG_1 and MG_2 is the bridge. Recalling that plumbers use a $5 : 12 : 13$ triangle to install $22.5°$ elbows, you construct one on half of the bridge, so the full bridge subtends a $22.5°$ angle at any point on the circle that contains this kill chord centered at the apex of the triangle, O_1. These gunners' greatest fear is a BTR-80 on hill #1, so grenadiers are positioned along the riverbank aiming for this hill. MG_3 can join this fight if the enemy delays rushing the bridge.

The kill circle of MG_3 and MG_4 is the bridge exit. MG_3 is at the side of the store and aimed for point O_2; its field of fire is centered on the ray to O_2. Drop a perpendicular from O_2 to one ray of this angle and then draw the kill circle. MG_4 is given to the best marksman and concealed behind hill #3 where it fires on the kill circle of MG_3. Bisect the segment from it to point O_2 and then draw an arc centered at this midpoint and passing through O_2 to find the points tangent to the kill circle. Draw rays to these touching points to determine how much MG_4 is to traverse. The grenadiers near the bridge are below the MG_1 and MG_4 fire grazing the bridge; also, they are below the -4° minimum elevation angle of the BTR-80 cannon. They are fighting Apache style![6]

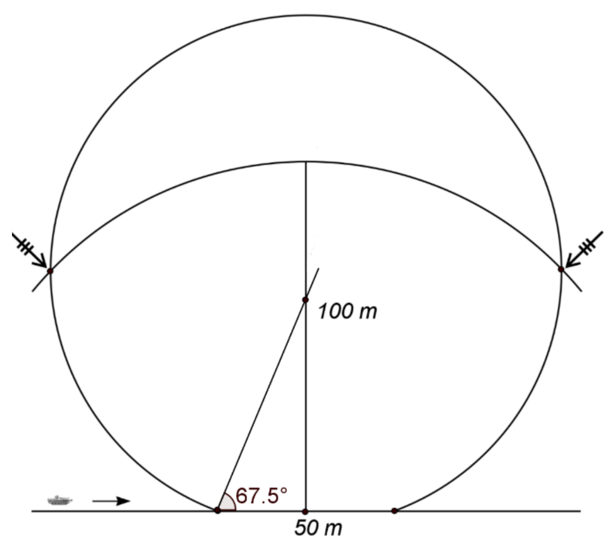
[6] Victorio's ambush at Cooke's Canyon on 29 May 1880 in New Mexico, USA is described by Watt (2012, p. 38).

The BTR-80 suspension was not designed by Lotus; speed bumps can really mess up their return fire! Concrete takes weeks to cure enough to stop 14.5 mm fire; speed bumps need only days. If the enemy comes in days, not weeks, do not make bunkers; install speed bumps on the bridge.

In this scenario, we took the top traverse to be 22.5° because, conveniently, this is an angle that I have already provided plumbers with instruction on how to construct. Protractors are forbidden in this textbook because real numbers have not been defined. But, since soldiers will typically read of the top traverse in their weapon's manual rather than be shown it, we will here allow the use of protractors for replicating and adding angles, just not multiplying them. If you subtract the top traverse from 90° and lay this angle off the endpoint of the kill chord with a protractor, the ray intersects the mediator at the center of the circle that contains the kill chord.

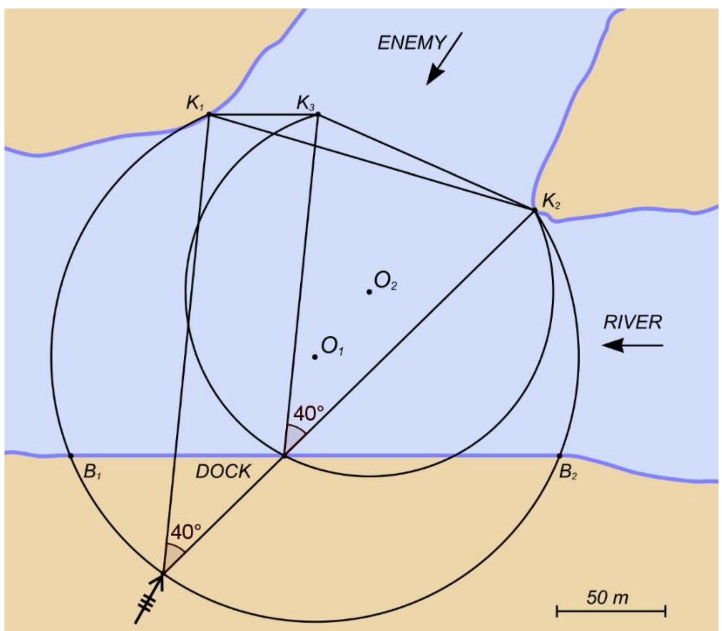
Scenario #2: We have four machine guns and wish to set an ambush from defilade, leading a BMP-2 that will drive across a 50-meter opening where vision is obscured on either side. Knowing that the 12.7 mm M2 machine gun is marginal against vehicles heavier than the BTR-80, the armorer has mounted them in coaxial pairs, so the bullets converge at a range of 100 meters.

Thus, there are two criteria for the emplacement of our two dual guns. We wish them to sweep the given kill chord within their given top traverse and we *also* wish their nearly coaxial fire to converge at the midpoint of the kill chord so the bullets striking the same spot at the given distance might penetrate armor plate that the 12.7 mm is not actually rated for. We are constructing a pair of triangles given the base, the apex angle, and the median to the base.



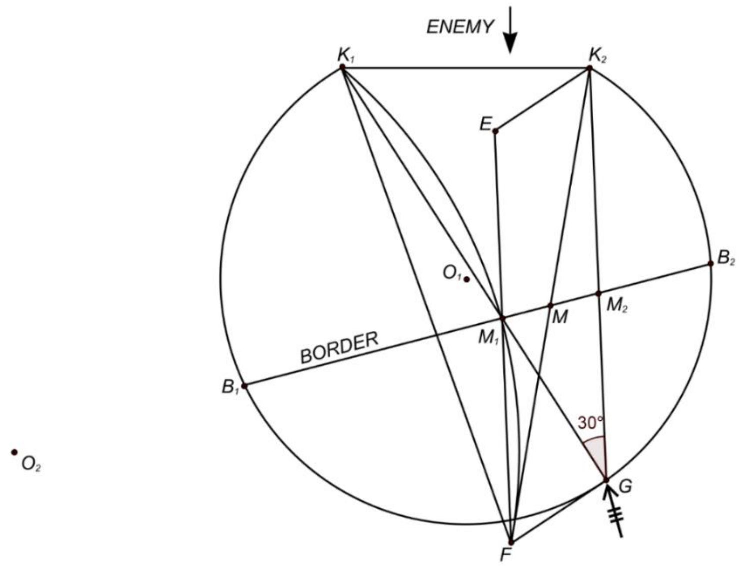
Scenario #3: Here we are fighting either in the past or in a modern country too poor to have any armored vehicles nor many machine guns. The border is defined by a river. At the confluence of it and another river that comes from your enemy's land, you, with only one machine gun, are tasked with defending against enemy marines who might come in small boats and attempt to cross the river under fire. On your bank of the river, you have much concertina wire. If you can kill half the enemy with your gun, your infantry can fight off the other half at the wire. However, during peacetime you trade with these people, and so there must be a 50-meter-wide break in the wire for a dock. To protect your gun from mortars, you will build a bunker with a 40° slit.

The soldiers wish the dock were narrower and the traders wish it were wider, so it must be *exactly* 50 meters to satisfy both interests. But it can be built anywhere on the chord defined by where your riverbank passes through the circle defined by the kill chord, which is defined by the width of the enemy river and by the 40° top traverse of the gun. The objective is to position it so the army's only machine gun can perform double duty: It must cover the kill chord across the enemy river, and it must *also* cover the dock that they hope to storm.



$\overline{K_1K_2}$ is the kill chord and O_1 is the center of the circle defined by the intersection of the kill chord mediator and a 40° angle laid against the kill chord. To avoid clutter, this triangle is not shown. B_1 and B_2 are the points on the riverbank between which your gun must be positioned on the O_1 arc to cover the kill chord. From K_1 draw a line parallel to $\overleftrightarrow{B_1B_2}$. Towards K_2, lay off a segment 50 meters long to K_3. This is one side of a parallelogram with the dock on the other side, but we still do not know any of the parallelogram's angles and so it cannot yet be drawn. By the transversal theorem, the two angles shown are equal and so $\overline{K_2K_3}$ also subtends a 40° angle. Find O_2 the same way O_1 was found and draw this circle. Where it intersects $\overleftrightarrow{B_1B_2}$ is the K_2 side of the dock. Circles typically intersect lines twice, as this one does, so the dock could have been positioned with its endpoint at the other intersection, but we chose to enfilade the enemy river.

Scenario #4: A straight fence defines the border between two countries, and, on the enemy side, geographic features define a bottleneck. Given a top traverse of $30°$, the kill chord drawn across the bottleneck defines an arc that your gun must be positioned on. There is a point on the fence, M, that one infantry platoon is tasked with defending to the left of and another platoon to the right of. To avoid the appearance of favoring one platoon over the other, you wish to position your gun so its field of fire covers equal segments of fence to the left and to the right of this point.



$\overline{K_1K_2}$ is the kill chord and O_1 is the center of the circle defined by the intersection of the kill chord mediator and a $30°$ angle laid against the kill chord. To avoid clutter, this triangle is not shown. B_1 and B_2 are the points on the border fence between which your gun must be positioned on the O_1 arc to cover the kill chord. Guess where the gun, G, is to be positioned and draw in its field of fire. Position M on the fence midway between the edges of the field of fire. We will draw a figure and then learn from it so we can draw it again starting with M at its given position and then finding G, rather than starting with G and then finding M, which we did for the first drawing.

If M is the bi-medial point of a parallelogram, $\overline{EFGK_2}$, then, by the parallelogram centroid theorem, it bisects any segment from one side to the other, including the segment of fence in the gun's field of fire, $\overline{M_1M_2}$, and the diagonal, $\overline{FK_2}$. Extend $\overrightarrow{K_2M}$ an equal distance to find F and then draw in the rest of $\overline{EFGK_2}$. By the transversal theorem, $\angle EM_1K_1$ is $30°$ and its supplement, $\angle FM_1K_1$ is $150°$. Thus, $\overline{FK_1}$ could be the kill chord for a second gun with the same top traverse of $30°$ positioned on an arc centered at O_2. Now let us redraw our figure with M in its given position. Extend $\overrightarrow{K_2M}$ an equal distance to find F; we cannot draw $\overline{EFGK_2}$ because we have only the diagonal. Connect $\overline{FK_1}$, find O_2 and then draw in the part of the O_2 circle on the other side of $\overleftrightarrow{FK_1}$. Where it intersects $\overline{B_1B_2}$ is M_1. Extend $\overrightarrow{K_1M_1}$ to find G on the O_1 circle. Connect $\overline{GK_2}$.

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