

Theoretical Foundation for the Inverse Power Law Distribution

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Abstract

It has come to my attention that a solution to the puzzle of why large fluctuations in prices have an inverse power law distribution goes unanswered. This is an easy corollary to the principal result of my 1999 book, **Axiomatic Theory of Economics**, Theorem 12, the Law of Price Adjustment, summarized in this paper.

My theory describes a single instant in time. Of course, instants have the habit of following one another, eventually forming something called history. So it is natural to inquire, not just what the price is at a given instant, but what the distribution of price changes is over history. This I did not do in 1999.

Addressing this issue requires an additional axiom: the parameter called importance, μ , must have an exponential distribution, $\lambda e^{-\lambda\mu}$. Specifically, the inverse cubic law requires that the underlying distribution be $2e^{-2\mu}$.

In point of fact, the only restriction that I placed on μ is that it be a non-negative real number. But negative monotonic distributions with this support and moments of all orders are not that numerous. Why not the exponential? The fact that an exponential distribution of μ implies an inverse power law distribution of large price fluctuations is motivation enough for most economists to accept this new axiom, and my other results require only that the distribution of μ be negative monotonic on $[0, \infty)$.

Keywords

Axiomatic Economics, Law of Price Adjustment, Econophysics, Power Law, Inverse Cubic Law, Inverse Cubic Distribution, Distribution of Price Fluctuations

there are a fixed number of definitions in each series of similar definitions which have phenomena conforming to them. Marginal utility is defined with the quantity of phenomena conforming to each definition, not an arbitrarily fixed constant, but a function of wealth.

I assert that one is capable of determining which of any two phenomena or sets of phenomena conform to a definition at a higher place on one's value scale than the other. If one fails to determine which of the two is higher, it can only mean that they are equal. In other words, one's value scale is a total (linear) ordering of phenomena. This is the first of three axioms which the reader is asked to accept. The plausibility of this axiom is derived mainly from analogy with the other dimensions (space and time), which are also totally ordered. A total ordering is included in the assertion of Absolute Geometry that every line has a coordinate system.

Because of this axiom, for every definition on one's value scale to which phenomena might conform, there stands beside it the number of units of money to which one is indifferent as to which one received. This supposition demands only that money be infinitely divisible, which it is for all practical purposes. As there are an infinity of distinct points on one's value scale, however, it cannot be expected that one is conscious of them all. In fact, one does not need to know exactly what one's point of indifference is to conduct many transactions.

The graph of the distribution of points of indifference, $c(m)$, can be pictured as an aerial view of the people who value a phenomenon assembled along a line marked "money", where they are asked to stand by the number of monetary units that are equal to a unit of that phenomenon. If more than one person has the same valuation, they stand behind the corresponding number. The stock of that phenomenon naturally tends toward the high end, as anyone who possesses a unit of it who sees his neighbor to the right without one will sell it to him. Only use-value and expected exchange-value in other markets not represented on this graph are counted because, though one may value a phenomenon greatly in anticipation of exchanging it at a high price, if one fails to get that price, one has to lower one's asking price until it eventually equals the value of keeping that phenomenon for one's personal use. While money has very little use-value, it does have expected exchange-value in other markets not represented on this graph, and it is with this in mind that people withhold their money from this market if the price rises too high. The

expected exchange-value of money is historically derived from its use-value. If it were a function of today's prices, we would have a contradiction because we are now deriving today's prices from the demand distribution, $c(m)$, which includes expected exchange-value.

If a phenomenon has a steeply-diminishing utility for most people (after acquiring one unit, the importance of the next is very low because one easily becomes sated), most people are only represented once and $c(m)$ is very close to $c_0(m)$, the distribution of people's point of indifference for their first unit. If there is a gradually-diminishing utility among people, many come back again and again before they become sated, each time with a lower point of indifference, and consequently the low end of $c(m)$ rises.

$R = \int_0^{\infty} c(m) dm$ is the requirement for a phenomenon by a population. Because

stock is limited, however, only those with the highest use-value of it relative to their value of money possess any of the phenomenon. The price is less than the point of indifference of the last person who possesses a unit of the phenomenon or he would sell it, and it is greater than the point of indifference of the first excluded individual or he would buy. These two points of indifference are the marginal pair which determine the upper and lower limit of price, between which is the zone of indeterminacy. The formula relating

price and stock to the demand distribution, $c(m)$, is $S(m) = \int_m^{\infty} c(t) dt$ with $S(m)$

the stock, m the price, and $c(t)$ (t is a dummy variable for the integration) the distribution of points of indifference between the use-value of a unit of a phenomenon and t units of money. Of course, the expression above does not have any meaning until it is proven that stock converges. It will be used informally, however, until the convergence of stock is proven.

Both the people traditionally labeled "consumers" and those labeled "producers" appear in the demand distribution. The conceptual separation of consumers and producers is a great mistake of mainstream economics. They are all just people, each with a bit of the stock, and they are all prepared to sell if the price is above a certain point and buy if the price is below that point. The only thing that distinguishes people from one another is their point of indifference. This has little to do with who produced different bits of

the stock, the event of production having occurred in the forgotten past. When economists draw one curve called "supply" and another called "demand", they are implying that the two are independent, for one cannot solve two simultaneous equations for two variables if the two equations are just versions of the same relation. Their dependence is well known at the macro level, but I assert that supply and demand are not independent at the micro level either. It is a mistake to inquire whether I support Say's assertion that "supply creates its own demand" or Keynes' assertion that "demand creates its own supply"; **Axiomatic Theory of Economics** is detached from that debate. I anticipate that the greatest block to the understanding of my theory will be people trying to interpret it in terms of supply and demand. I do not believe in supply and demand. I believe in the demand distribution, which is a mapping between price and stock. Supply has no place at all in **Axiomatic Theory of Economics**. My theory is not even divided into "micro" and "macro" sections. These terms were invented by mainstream economists when it became necessary to paste Keynes' theory over the top of Marshall's theory. They are clearly incompatible and their association in modern textbooks is entirely due to the bookbinder, not the economist.

By what criterion does mainstream economics distinguish people represented on a supply curve from those represented on the associated demand curve? This is a particularly pressing question for people dealing in narcotics because the penalties are so much greater for being on one curve than the other. But, if one visits a neighborhood where such trade takes place, any of the people one encounters would sell if the price were right and would buy if offered a bargain. There is really only one relation and it is called the demand distribution. Since there are two variables, price and stock, this (single) relation can provide a mapping from one variable to the other but cannot fix them. However, later in this article, existence and uniqueness proofs are given for a point toward which price and stock tend. Thereafter, it will be assumed that they are fixed at that point, called saturation.

The method of mainstream economics really has a third variable which is never mentioned and that is the time unit for supply and demand. It is well known that elasticity is a function of this time unit and, if this is true, one calculates a different price depending on whether one speaks of weekly or monthly supply and demand. This is an inconsistency since there can only be

one price and it is not dependent on the caprice of an economist when he decides how often to conduct his surveys. This is a point that is glossed over in mainstream texts. A detailed discussion of the time unit chosen for supply and demand is never given and many texts neglect to mention the need for choosing one at all. Yet in their chapter on elasticity, every textbook lists time as a factor, sometimes as the most important factor.

Mainstream economists have two variables, price and quantity per unit of (some usually unspecified) time, and two equations, supply and demand. For this to work at all, the equations must be independent, which means that each individual must be either a buyer or a seller. The economist's decision to put people on one curve or the other cannot depend on the price that they would buy or sell because both equations are defined for all prices. (Price is one of the independent variables.) So what is the economist's decision based on? Ask him repeatedly until he admits that there is really only one distribution. Also, press him to acknowledge that the demand distribution independently exists at each instant of time. Supply and demand curves are different depending on the time unit chosen. Mainstream economists provide no proof that their predicted prices are independent of their choice of time unit. For example, will thirteen predicted weekly quantities be the same as three predicted monthly quantities?

A large part of the problem with supply and demand is that it is used descriptively, but called predictive. It is easy to predict the past. Economists just observe the quantity produced one month and what it sold for and they put a little \times over that spot. Then, by pure conjecture, they draw four tails on their \times to fill their graph paper. Supply and demand has never been used predictively, not even to make bad predictions. \times marks the spot is a purely descriptive technique. Since they are using the 20-20 vision of hindsight, they can do this for three months in a row and, to nobody's surprise, the sum of the quantities is the quarterly quantity. In the real world, price is constant for years at a time but, for most companies, their weekly and monthly sales figures swing wildly and unpredictably, sometimes by several fold from one month to the next. Mainstream economists have no explanation for this, which they should since their theory is called supply and demand and the horizontal axis of their graph is labeled weekly (or monthly) quantity. When I have been asked to help predict sales, I have told them that price is related to stock, not supply, and that they should stop watching their sales chart so

ardently. At most companies, there is someone in accounting who feeds sales figures to the employees so that they can predict layoffs. They know that every dip in sales will send hundreds of them to the unemployment office, and that every rise will have their bosses clapping each other on the back and extolling their brilliant and farsighted management. They also know that nobody can predict sales. Supply never means anything in economics, though sometimes (for non-durable phenomena) it can pass for stock.

It is well known that mainstream economics is in trouble. Nobody in the hard sciences respects economists and even within their own ranks, a number of books and articles have appeared questioning why economics is not yet a science. There is considerable debate among economists about methodology, what it takes to qualify as a science, and what distinguishes economics from other fields. Implicit throughout is the understanding that mainstream economics does not work. To qualify as a science, economics must be axiomatic. But one must address price and stock; supply and demand does not work. Also, to deduce mathematical expressions from axioms, the axioms must be of a mathematical nature and they must specify actual functions from which equations can be derived. Fortunately, however, there is nothing fundamental about economics that prevents it from being made into a science just as physics was made into a science by Newton and mathematics by Euclid. This is what I propose to do.

II

There is an upper bound to one's value of any stock of a phenomenon which will be denoted M . This includes one's need for saving phenomena for future use. Total utility is the marginal utility of a phenomenon when the unit is defined as the entire quantity possessed. It increases from zero up to its maximum point, M , as one's stock increases. Hence, total utility is a cumulative distribution function and marginal utility is the associated probability density function (after normalization), denoted $U(s)$ and $u(s)$, respectively. Since the utility of a given stock is measured by the quantity of money which stands beside it on one's value scale, $U(s)$ is a mapping from the stock of a phenomenon one possesses to the money one associates with that stock. $u(s)$ is its first derivative. $u(s)$ must be negative monotonic because utility diminishes as one adds units to one's stock. The integral of $u(s)$ must also converge, that is, $\int_0^{\infty} u(s) ds < \infty$. This is because marginal utility is the probability

density function of a cumulative distribution. Nothing else is known about $u(s)$ and this is the first parameter (and the only function) used to distinguish phenomena from one another. Its characteristics must be regarded as an axiom. Later, two more parameters (both from \mathfrak{R}^+) will be introduced which will be sufficient to completely describe every phenomenon.

As will be shown shortly, we are only concerned with the ratio $\frac{u(0)}{u(r)}$ for non-negative integers, r . This ratio is invariant under a re-scaling of the vertical axis, so $u(s)$ can be normalized by setting the upper bound on the distribution function, M , to unity. This makes $u(s)$ a true probability density function as the total area under it is unity.

It should be noted here that the requirement that $u(s)$ be negative monotonic does not imply that firms must be small, which is clearly not true because there are many large and successful corporations. Economists have used the term "marginal (or diminishing) utility" to denote both the first derivative of one's total utility for some phenomenon and the assertion that

firms receive less and less return on their investments as they grow bigger. Capital, like all phenomena, has diminishing utility because one quickly becomes sated on it. However, like most things on which one temporarily sates oneself, one is ready for more the next day and the day after that. Thus, while a firm cannot immediately make use of all the capital it might consider buying, it can start with a small capital project and use the profits from that to train the managers and laborers that will make an expansion feasible. In this way, firms can become global in scale without ever contradicting the assertion that $u(s)$ is negative monotonic for capital. The large corporation embarking on another great expansion may have started out as a small mom-and-pop outfit, but it is not that little company anymore and it has a (very) different utility function now. Since **Axiomatic Theory of Economics** is about stock, not supply, the relative sizes of the firms supplying a phenomenon is of no concern.

I assert that the distribution of people's points of indifference for their first unit of a phenomenon relative to money, $c_0(m)$, is lognormal; that is, the natural logarithm of the number of people who are indifferent at a particular price, m , is cumulatively (normally) distributed. The cumulative distribution is applicable to a variable that is subject to a process of change such that, at each step, a random quantity is added to the accumulated value of that variable. By the Central Limit Theorem, the distribution of the sum of a large number of independent, identically-distributed random variables (from an unspecified distribution with a finite mean and a non-zero, finite variance) is approximately normal. $c_0(m)$, however, does not accumulate, rather it is analogous to the growth of the value of money through history: It conforms to the characteristics of proportionate effect. After the j 'th day of a person's life, the change in the number of monetary units to which he is indifferent, relative to the first unit of a phenomenon, is a proportion of his indifference point the day before. That anthropometric variables (height, size of organs, tolerance to drugs, etc.) conform to the characteristics of proportionate effect is well established in the literature.

Theorem 1 (Law of Proportionate Effect): Phenomena which conform to the characteristics of proportionate effect are lognormally distributed.

Proof:

$$m_j - m_{j-1} = \varepsilon_j m_{j-1}$$

The difference between each step and the last one is the last one multiplied by a random quantity.

$$\frac{m_j - m_{j-1}}{m_{j-1}} = \varepsilon_j$$

Divide through by m_{j-1} to get ε_j , the change in m relative to its previous value, m_{j-1} .

$$\sum_{j=1}^n \frac{m_j - m_{j-1}}{m_{j-1}} = \sum_{j=1}^n \varepsilon_j$$

Find the sum of all ε_j from the initiation of the process to its termination after n steps.

$$\int_{m_0}^{m_n} \frac{dm}{m} = \sum_{j=1}^n \varepsilon_j$$

If each step is small, $m_j - m_{j-1}$ can be approximated by dm .

$$\ln|m_n| - \ln|m_0| = \sum_{j=1}^n \varepsilon_j$$

Integrate from m_0 to m_n .

$$\ln|m_n| = \ln|m_0| + \varepsilon_1 + \dots + \varepsilon_n$$

Solve for $\ln(m_n)$.

As can be seen from the last step, the natural logarithm of one's indifference point after the n 'th day is a constant (the logarithm of its initial quantity) with a large number of random and identically-distributed quantities accumulated onto it. Hence, after having lived through n days and having seen their point of indifference change by a small proportion each day, consumers of their first unit are normally distributed with regard to the variable $\ln(m)$ and, hence, are lognormally distributed with regard to the variable m . ■

The absolute value operation may be dropped, since we are only interested in positive prices.

That first-unit demand conforms to the characteristics of proportionate effect must be regarded as an axiom. A plausibility argument is provided here. Let $m_j = \phi(m_{j-1})$ with m_j the number of monetary units to which one is indifferent relative to the first unit of a phenomenon on the j 'th day of that person's life. We want to show that $\phi(m_{j-1}) = (1+\varepsilon_j)m_{j-1}$. Consider a man who wants to take out a loan at interest. He must think he will have more money in the future than he does now. (More money holdings, not necessarily more wealth.) If he does, the value of individual monetary units will tend to decrease over time relative to other phenomena; that is, ϕ is a positive function when averaged over all phenomena. To determine how much interest he is willing to pay, the man must specify this average ϕ . For him to calculate the interest owed per unit of time as a percentage of the principle is equivalent to specifying $\phi(m_{j-1}) = (1+\varepsilon)m_{j-1}$ with $\varepsilon > 0$ fixed. Fixing ε is a special case of ε_j being a random variable. Here, the probability density function is unity at ε and zero elsewhere. Thus, the axiom that first-unit demand conforms to the characteristics of proportionate effect is a generalization of calculating interest as a percentage of the amount owed. In fact, this is how people have calculated interest throughout recorded history, although economics having always been a soft science, they never asked for proof. Perhaps the value of money decays harmonically over time or in another way besides exponentially? This question is addressed in **Axiomatic Theory of Economics**, but for now let us proceed to investigate the consequences of people's points of indifference for their first unit of each phenomenon being lognormally distributed. I believe that this axiom is on solid intuitive ground and will not be criticized. Even if it is, it is unlikely that critics will succeed in convincing the banking industry to calculate interest with a different formula, so the weight of tradition will continue to support my choice of the lognormal distribution for first-unit demand.

Before continuing, let us explicitly state our three axioms:

- 1) One's value scale is totally (linearly) ordered:
 - i) Transitive; $p \leq q$ and $q \leq r$ imply $p \leq r$
 - ii) Reflexive; $p \leq p$
 - iii) Anti-Symmetric; $p \leq q$ and $q \leq p$ imply $p = q$
 - iv) Total; $p \leq q$ or $q \leq p$
- 2) Marginal (diminishing) utility, $u(s)$, is such that:
 - i) It is independent of first-unit demand.
 - ii) It is negative monotonic; that is, $u'(s) < 0$.
 - iii) The integral of $u(s)$ from zero to infinity is finite.
- 3) First-unit demand conforms to proportionate effect:
 - i) Value changes each day by a proportion (called $1+\varepsilon_j$, with j denoting the day) of the previous day's value.
 - ii) In the long run, the ε_j 's may be considered random as they are not directly related to each other nor are they uniquely a function of value.
 - iii) The ε_j 's are taken from an unspecified distribution with a finite mean and a non-zero, finite variance.

$\ln(m)$ is linearly transformed by $\frac{\ln(m)-\mu}{\sigma}$. The location parameter, μ (mean), quantifies the importance of a phenomenon relative to money and the scale parameter, σ (standard deviation), quantifies the difficulty of substituting other phenomena for the one in question. Easily-substituted phenomena have very little probability in the tail of their demand distribution; only the eccentric purchase a phenomenon at a high price when there are cheaper substitutes available. As substitution becomes more difficult, people must purchase the phenomenon even at high prices, and their distribution is less skewed. Both μ and σ must be positive. With $u(s)$, μ and σ describe all phenomena. Thus, every phenomenon is associated with a point in $u(s), \mu, \sigma$ space where $u(s)$ is a negative-monotonic probability density function on \mathfrak{R}^+ and μ and σ are both from \mathfrak{R}^+ . For the purpose of economics, nothing else distinguishes one phenomenon from another.

The equation for the distribution of first-unit demand is

$$c_0(m) = \frac{e^{-\frac{1}{2}\left(\frac{\ln(m)-\mu}{\sigma}\right)^2}}{\sigma m}. \quad \text{This is the equation of the lognormal distribution,}$$

$e^{-\frac{\ln^2(m)}{2}}$, multiplied by the derivative of the linear transformation which is substituted for $\ln(m)$. It need not be divided by its total area, $\sqrt{2\pi}$, since it will not be used as a probability density function.

The number of people with a point of indifference at a particular price, m , for their first unit is $c_0(m)$. Whoever's point of indifference for his first unit is $\frac{u(0)}{u(1)}$ times greater than that price values his second unit equivalent to

price m . Whoever's point of indifference for his first unit is $\frac{u(0)}{u(2)}$ times greater

than that price values his third unit equivalent to price m , and so on. To find $c(m)$, all the people with a point of indifference at m are summed up, whether it is their first purchase or a later purchase. Recall the analogy of the demand distribution being an aerial view of the people who value a phenomenon assembled along a line marked "money", where they are asked to stand by the number of monetary units that are equal to a unit of that phenomenon. Now consider a person who wishes to possess more than one unit of the phenomenon; each of his agents appears behind a different point on the money line. If he himself appears in the column assembled behind m monetary units, the first person he sends to get another unit is directed to the column behind $\frac{mu(1)}{u(0)}$ monetary units. His next agent is in the column behind $\frac{mu(2)}{u(0)}$ monetary units, and so on. Hence, we have the following formula for the demand distribution which, unfortunately, is impossible to integrate in closed form, even with $u(s)$ fixed.

$$c(m) = \sum_{r=0}^{\infty} c_0(x) \quad \text{with } x = \frac{mu(0)}{u(r)}$$

$c_0(m)$ can be thought of as the 0'th partial sum of $c(m)$ and, in general, $c_n(m)$ denotes the n'th partial sum of $c(m)$. Thus,

$$c_n(m) = \sum_{r=0}^n c_0(x) \quad \text{with } x = \frac{mu(0)}{u(r)}$$

Most of the real analysis in **Axiomatic Theory of Economics** stems from the infinite summation, $c(m)$. To simplify the proofs in this article, the second axiom is replaced with the assertion that people never need more than one of anything at a time. This assumption is neither accurate nor necessary, as all of the results of my theory can be (and are) proven in their full generality. However, some economists do not have the mathematical background necessary to read **Axiomatic Theory of Economics**, so, for expository purposes, simplified proofs are provided here. Also, before the theory becomes accepted, it will receive cursory reviews, perhaps at the end of courses on mainstream economics. In this case, if a professor is sympathetic to my theory, he may wish to prove some of its assertions, but he will not have time to prove them in their full generality. As long as he mentions that the complete proofs do exist, his students can get the essence of my theory from the simplified proofs. The important thing for them to understand is that this theory is deduced from axioms. So, for the remainder of this article, all of the theorems will be proven using the 0'th partial sum, $c_0(m)$, rather than $c(m)$. When $S(m)$ appears in a proof, it will refer to $S(m) = \int_m^{\infty} c_0(t)dt$. $f(\mu, m)$, which will be defined later, will also be defined in terms of $c_0(m)$ rather than $c(m)$.

III

Theorems are numbered analogous to those in **Axiomatic Theory of Economics**.

Theorem 4: $\lim_{m \rightarrow 0^+} c_0(m) = 0$

Proof: $c_0(m) > 0$ for all $m > 0$. Thus, by the Squeezing Theorem, if $c_0(m)$ is less than some function for all $m > 0$ and that function is continuous and equals zero at zero, then $\lim_{m \rightarrow 0^+} c_0(m) = 0$. Consider hm with h a finite constant. Since hm vanishes at zero, it is sufficient to show that

$$hm > \frac{e^{-\frac{1}{2}\left(\frac{\ln(m)-\mu}{\sigma}\right)^2}}{\sigma m} \text{ for all } m > 0.$$

By making the substitution $y = \ln(m)$, this is equivalent to $y^2 + (4\sigma^2 - 2\mu)y + 2\sigma^2 \ln(\sigma h) + \mu^2 > 0$ for all real y . By the Quadratic Theorem, this is true for $\frac{e^{2(\sigma^2-\mu)}}{\sigma} < h < \infty$. Thus, the demand distribution is equal to zero at zero. ■

Alternate proof: Make the substitution $y = \ln(m)$ so

$$\begin{aligned} \lim_{m \rightarrow 0^+} c_0(m) &= \lim_{y \rightarrow -\infty} \frac{e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}}{\sigma e^y} \\ &= \frac{1}{\sigma} \lim_{y \rightarrow -\infty} e^{-\frac{(y-\mu)^2 + 2\sigma^2 y}{2\sigma^2}} \\ &= \frac{1}{\sigma} \lim_{y \rightarrow -\infty} e^{-\frac{(y-\mu+\sigma^2)^2 - \sigma^2(\sigma^2 - 2\mu)}{2\sigma^2}} \\ &= 0 \quad \blacksquare \end{aligned}$$

The former was chosen as the main proof because the Squeezing Theorem and the Quadratic Theorem can be visualized and are (hopefully) more intuitive than a purely algebraic proof.

Theorem 7: Stock is finite.

Proof: Make the substitutions $y = \frac{\ln(t)-\mu}{\sigma}$ and $dy = \frac{dt}{\sigma t}$ so

$$S(m) = \int_z^{\infty} e^{-\frac{y^2}{2}} dy \quad \text{with } z = \frac{\ln(m)-\mu}{\sigma}$$

The integral is the standard normal distribution, which is tabulated as $\alpha(z) = 1 - \Phi(z)$ in the back of any statistics text, though multiplied by the constant $\frac{1}{\sqrt{2\pi}}$ so that the total area under the integrand is unity, a step which is omitted here since the integrand is not being used as a probability density function. However, since this integral never exceeds $\sqrt{2\pi}$, we have the following inequality: $S(m) < \sqrt{2\pi}$. ■

Aggregate utility is defined as price multiplied by stock. This is because money is the measure of utility and everyone who possesses a unit of stock values it only as highly as its replacement cost, for that is all that one risks. Stock and price, however, are inversely related, so increasing one or the other does not necessarily increase aggregate utility. Aggregate utility being the common goal of people dealing in a phenomenon, they are interested in maximizing it. As stock increases, aggregate utility also increases up to saturation, where any further increases in stock reduce aggregate utility by driving the price down. That part of the demand distribution to the right of saturation (the high end), where increases in stock increase aggregate utility, is unsaturated and that part to the left (the low end) is saturated. At a constant stock, there is a zone of indeterminacy between the marginal pair within which the price may fluctuate. Such fluctuations appear to be of a saturated market whether the stock has reached saturation or not. Most markets are large enough, however, that the zone of indeterminacy is too narrow to be of practical concern.

Because the actions appropriate in an unsaturated market (increasing stock) are not those appropriate in a saturated market (decreasing stock), it is important to determine the point of saturation. Aggregate utility, $mS(m)$, is at a relative maxima where its first derivative, $S(m) - mc(m)$, equals zero. Thus, saturation is a price and stock such that $S(m) = mc(m)$. Graphically, $S(m)$ is represented by the area between the horizontal axis and the graph of the demand distribution from m to ∞ . $mc(m)$ is represented by the area of the rectangle formed by the two axes and horizontal and vertical lines extending from the point $m, c(m)$.

Theorem 10 (existence): The absolute maximum of aggregate utility is at a finite critical point.

Proof: By Theorem 4, the limit of $c_0(m)$ at zero is zero. Thus, stock is finite even if it is free, and aggregate utility goes to zero as price approaches zero. Since aggregate utility is always positive, it is sufficient to show that it also goes to zero as price approaches infinity to prove the existence of a relative maxima. One makes the substitutions $y = \frac{\ln(t)-\mu}{\sigma}$ and $dy = \frac{dt}{\sigma t}$ so

$$\begin{aligned}
 0 < mS(m) &= m \int_{\frac{\ln(m)-\mu}{\sigma}}^{\infty} e^{-\frac{y^2}{2}} dy \\
 &\leq m \int_{\frac{\ln(m)-\mu}{\sigma}}^{\infty} ye^{-\frac{y^2}{2}} dy && \text{if } m \geq e^{\mu+\sigma} \\
 &\quad \frac{\ln(m)-\mu}{\sigma}
 \end{aligned}$$

$$\begin{aligned}
&= -m \left[e^{-\frac{y^2}{2}} \right]_{\frac{\ln(m)-\mu}{\sigma}}^{\infty} \\
&= me^{-\frac{1}{2} \left(\frac{\ln(m)-\mu}{\sigma} \right)^2} \\
&= me^{-\frac{\ln^2(m)-2\mu\ln(m)+\mu^2}{2\sigma^2}} \\
&= \frac{Bme^{-\frac{\ln^2(m)}{2\sigma^2}}}{m^{-\frac{\mu}{\sigma^2}}} \quad \text{with } B = e^{-\frac{\mu^2}{2\sigma^2}} \\
&= \frac{Bm}{m \frac{\ln(m)-2\mu}{2\sigma^2}} \\
&\leq \frac{B}{m} \quad \text{if } m \geq e^{2(\mu+2\sigma^2)}
\end{aligned}$$

B is a constant, so $\lim_{m \rightarrow \infty} B/m = 0$ and, by the Squeezing Theorem, $\lim_{m \rightarrow \infty} mS(m) = 0$. Thus, there exists a finite price where aggregate utility is at a maximum. ■

This only proves the existence of a relative maxima and identifies it with the absolute maximum. There may be more than one relative maxima, in which case the largest of them is the absolute maximum. However, by the following proof there is only one relative maxima and it is the absolute max-

imum of aggregate utility. This justifies the use of the word "the" when referring to the saturation point.

Theorem 11 (uniqueness): Aggregate utility has only one relative maxima.

Proof: Because aggregate utility is always positive and it approaches zero at both ends of its domain, $(0, \infty)$, there is either a single relative maxima, or relative maximas and minimas alternate with the largest and smallest being relative maximas. The second derivative of aggregate utility is $c_0(m) \left(\frac{\ln(m) - \mu}{\sigma^2} - 1 \right)$. It is positive at relative minimas and negative at relative maximas. Therefore, if there is more than one relative maxima, there are two disjoint intervals in $(0, \infty)$ where the second derivative is negative and they are separated by an interval where the second derivative is positive.

We wish to show where the second derivative is strictly negative. $c_0(m) > 0$ for all m , so we only have to examine $\frac{\ln(m) - \mu}{\sigma^2} - 1$. This is negative for all $0 < m < e^{\mu + \sigma^2}$ and positive for all $m > e^{\mu + \sigma^2}$. Recalling that relative maximas and minimas alternate with the largest and smallest being relative maximas, there can only be one price such that $S(m) = mc_0(m)$ and it is a relative maxima. ■

It is an easy corollary that the saturation price is less than $e^{\mu + \sigma^2}$.

μ and σ change over time for a variety of reasons, each change necessitating a recalculation of the saturation point. It is the business of entrepreneurs to anticipate these changes and to adjust stocks accordingly. While most shifts in a demand distribution are of only local concern, one is of particular interest to economics. If some of the people represented by the demand distribution for a phenomenon receive money from the government, how does the saturation point change? Whether these people receive a grant, a low interest loan, or are doing contract work for the government, they are more liquid than they want to be. Knowing the negative effect of a loose monetary policy on the value of money, they are not going to hoard it. Relative to money, the importance of phenomena has increased. How are prices and stocks affected and which adjusts more dramatically to the increase in μ ?

It is an old adage that people get more out of something the more they put into it and, money being the measure of utility, one expects increases in the importance of a phenomenon relative to money to increase the phenomenon's price in proportion to the price that it has already attained. Mathematically, $p = p_0 e^{\mu}$, with p_0 the price at saturation with no importance relative to money and p the price such that $f(\mu, m) = S(\mu, m) - mc(\mu, m) = 0$. Notice that p is the particular price which satisfies the condition $f(\mu, m) = 0$ while m denotes an arbitrary price. Variables included in the functional notation are allowed to vary while others which appear in a function but are not listed in the parenthesis of the function are assumed to be constant. Here, we are discussing changes in both price and importance where before only price was allowed to vary.

Theorem 12: The price at saturation increases exponentially in response to an increase in the importance of a phenomenon relative to money; that is, $\frac{dp}{d\mu} = p$.

Proof: $f(\mu, m) = S(\mu, m) - mc_0(\mu, m) = 0$ implicitly defines a level set in the μ, m plane. Let that level set be parametrized by $[\mu(t) \ m(t)]$. By the chain rule, the derivative of $f(\mu, m) = 0$ is $\frac{\partial f}{\partial \mu} \frac{d\mu}{dt} + \frac{\partial f}{\partial m} \frac{dm}{dt} = 0$ or $\left[\frac{\partial f}{\partial \mu} \ \frac{\partial f}{\partial m} \right] \left[\frac{d\mu}{dt} \ \frac{dm}{dt} \right] = 0$. The latter vector is the derivative (tangent) of the parametrized level set, so $\left[\frac{\partial f}{\partial \mu} \ \frac{\partial f}{\partial m} \right]$ is perpendicular to the level set which passes through any μ, m where it is evaluated. From the definition of saturation, this is downward (toward smaller m), so a 90° counter-clockwise rotation of $\left[\frac{\partial f}{\partial \mu} \ \frac{\partial f}{\partial m} \right]$ is tangent to the level set of all μ, m combinations with $f(\mu, m)$ constant. Dividing its vertical component by its horizontal component gives the desired rate of change in price:

$$\frac{dm}{d\mu} = -\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial m}} = \frac{m \frac{\partial f}{\partial m}}{\frac{\partial f}{\partial m}} = m \quad \text{with } \frac{\partial f}{\partial m} = c_0(m) \left(\frac{\ln(m) - \mu}{\sigma} - 1 \right)$$

This relation is true regarding the level set which passes through any point μ, m . Choosing only points along the level set $f(\mu, m) = 0$ (rather than another constant) yields $\frac{dp}{d\mu} = p$. ■

Notice that $f(m)$ in the above proof may be expressed as

$$f(m) = -\int_m^{\infty} f'(t) dt = \int_m^{\infty} c_0(t) \left(1 - \frac{\ln(t) - \mu}{\sigma^2} \right) dt$$

Also, the evaluation of $\frac{\partial f}{\partial \mu}$ requires an application of Leibnitz' Rule, justification of which is given in **Axiomatic Theory of Economics**. Incidentally, it does not matter that the rotation is counter-clockwise since a clockwise rotation also switches the components but negates $\frac{\partial f}{\partial \mu}$ instead of $\frac{\partial f}{\partial m}$. Because the sign comes out front after the division, it is immaterial which way $\left[\begin{array}{cc} \frac{\partial f}{\partial \mu} & \frac{\partial f}{\partial m} \end{array} \right]$ is rotated.

An alternative proof uses the chain rule to differentiate $f(\mu, g(\mu, m)) = 0$ with $p = g(\mu, m)$ to get

$$f_{\mu}(\mu, g(\mu, m)) + f_m(\mu, g(\mu, m))g_{\mu}(\mu, m) = 0$$

This equation is solved for $\frac{dp}{d\mu} = g_{\mu}(\mu, m)$. Notice that, by the uniqueness of saturation, $p = g(\mu)$ is a function; that is, a unique price is associated with every μ , though in general this is not required for $g_{\mu}(\mu, m)$ to be determined explicitly. In other words, not every $g_{\mu}(\mu, m)$ has an anti-derivative, $g(\mu)$. By

the construction of $g_\mu(\mu, m)$, $g(\mu, m)$ is proven to be smooth and continuous, which is all that is required of it.

Until this proof, only one semester of calculus had been required of the reader. Theorems 12 and 13 are about functions of two variables, however, and are more difficult. Readers with only one semester of calculus may find the alternative proof of Theorem 12 easier than the main proof if they are familiar with implicit differentiation. However, many students who have been introduced to calculus of several variables readily grasp the concept of level sets because of their familiarity with contour maps. Thus, for $f: \mathcal{R}^{1+1} \rightarrow \mathcal{R}^1$, recourse to the tangent seems more intuitive than a purely algebraic proof and the former was chosen as the main proof. Readers with only one semester of calculus can obtain most of the mathematics they need by reading a textbook on multivariable calculus up to but not including Lagrange multipliers. This is generally considered the easy part of multivariable calculus and is the work of six or eight lecture hours. To read **Axiomatic Theory of Economics** (without the simplifying axiom of this article) also requires some knowledge of infinite series. Fortunately, the “hard” part of multivariable calculus (multiple integrals and vector fields) is never used. **Axiomatic Theory of Economics** is similar to probability. Indeed, I see my book following in the tradition of Kolmogorov’s **Foundations of Probability** more than in any work of an economist. People who have worked with probability distributions are encouraged to read **Axiomatic Theory of Economics** even if they are only vaguely familiar with multivariable calculus.

By Theorem 12, the price at saturation increases exponentially in response to an increase in the importance of a phenomenon relative to money. What about stock? Intuitively, one expects stock to remain constant since, effectively, all the government does by issuing money is to change the figures in which prices are quoted and that should not affect the stock of phenomena that people keep in existence. Most economists would agree that this is true in the long run but would argue that, because of the uneven diffusion of fresh issues of money, the stock of phenomena is temporarily affected. Money diffuses unevenly from a central bank and that is the principal motivation for issuing it (otherwise those close to a government would not profit from their connections), but I assert that this does not provide any incentive for the stock of phenomena to increase.

Theorem 13: The stock at saturation remains constant in response to an increase in the importance of a phenomenon relative to money; that is, $\frac{dS_p}{d\mu} = 0$.

Here, the subscript on stock denotes that it is the stock associated with the saturation price, p .

Proof: We are interested in the change in stock along the level set implicitly defined in the μ, m plane by the relation $f(\mu, m) = 0$. As noted in the preceding proof, the tangent to this curve is $[1 \ m]$. Normalizing this vector and taking the inner-product with the derivative of $S(\mu, m)$ gives the desired rate of change in stock. Since we are interested in proving that this change is always zero, it is sufficient to show that the numerator is always zero and we may omit normalizing the directional vector. The inner product of this with the derivative of stock, $[mc_0(\mu, m) \ -c_0(\mu, m)]$, is zero. ■

Together, the two preceding theorems will be referred to as the Law of Price Adjustment. Because Theorem 13 is a corollary of Theorem 12, the term "Law of Price Adjustment" is used to denote both theorems. From a practical point of view, however, the assertion that the stock of phenomena is unaffected by depreciating a currency is more important because, by definition, economics is concerned with the wealth of a nation. Of course, the wealth of an individual can always be increased at the expense of other people by printing and spending money, but theoretical economics (hopefully) addresses more lofty aims.

It is important that the Law of Price Adjustment does not place any restrictions on marginal utility, on the importance of a phenomenon relative to money, or on the difficulty of substituting other phenomena. Within my economic theory, these three characteristics are all that distinguish phenomena from one another; that is, phenomena with the same $u(s)$, μ , and σ are isomorphic. Thus, it is impossible to argue that my theory is inapplicable in certain situations because it has been proven to apply to all possible situations; that is, it applies to phenomena at every point in $u(s), \mu, \sigma$ space. Since any mathematician will confirm the deduction of the Law of Price Adjustment from the three axioms, for an economist to accept or reject the Law of Price Adjustment is equivalent to his acceptance or rejection of the three

axioms, respectively. Attempts to divert the argument away from the acceptance or rejection of the theory's axioms should be discouraged.

The implications of the Law of Price Adjustment should be obvious to anyone who has studied mainstream economics; stickiness of prices is the cornerstone of Keynesian Economics. Even for those who do not follow the mathematics, common sense alone is sufficient to refute the Keynesian premise. Considering that a government can print money for itself within a day's notice, if the adjustment process could not be done in equal time, the whole system of indirect exchange would have collapsed long ago. Prices can be changed with a word, but the stock of phenomena can only be changed after considerable toil. It is obvious which is adjusted and which left constant. The average level of prices is "sticky" because it takes time for money to diffuse through a community and if one is averaging all prices, it is some time before one notices a change in one's statistics. This average is also meaningless for the same reason. The effect of issuing money is to redistribute wealth to the people who receive the new money first and that is only possible because of the slow diffusion of money through an economy.

Having arrived at a position so fundamentally opposed to mainstream economics, it is important to realize exactly where we parted company. The difference is that my theory is concerned with the price and stock of phenomena while mainstream economics is concerned with the price and supply of phenomena. I assert that the stock of phenomena is more important than the supply because all of the decisions made regarding a phenomenon are based on its stock (how much of it is in existence), and not on how much of it happened to be produced in some arbitrary time period. Phenomena are the same whether they are produced in one time period or another. Most people do not know and none care what the supply of phenomena is, they are concerned with the stock; this week's or month's supply is only a small part of the available stock. Even if a factory is temporarily closed for a week or a month, the price of its product is hardly affected because the total amount of phenomena in existence is hardly affected. Yet during that week or month the supply is zero. Mainstream economics, which relates price to supply, is unable to explain why the price does not increase dramatically as inspection of the supply and demand curves predicts that it should.

Parking on campus has a price, so mainstream economists must believe that there is a supply, that is, an influx, of parking spaces. Yet none are being

produced. Clearly, it is the stock, the absolute quantity of them, that determines price. Supply never means anything in economics, though sometimes (for non-durable phenomena) it can pass for stock. There are three principle mistakes of mainstream economics, but addressing supply and demand instead of price and stock is the most egregious. The other two are assuming that all short-term credit instruments function as money and believing that the average price level is a meaningful statistic and, hence, that prices are “sticky.”

2013 Addendum

All of the preceding text is from the introduction to my 1999 book, **Axiomatic Theory of Economics**. At that time I did not consider the distribution of price fluctuations and was unaware that any empirical work had been done on this question. The manuscript was accepted by the publisher in 1996 before the work of Gopikrishnan, *et al* (1998), which econophysicists are now so proud of in spite of the fact that Buchanan (2013) makes it clear that they still do not have any explanation for these statistics.

By Theorem 12, $p = p_0 e^\mu$, with μ being any non-negative real number, so prices could be any amount greater than or equal to p_0 . But let us now propose an additional axiom and prove a theorem with it:

Axiom 4: The distribution of importance, μ , is exponential.

Theorem 14: The distribution of large price fluctuations (relative to the minimum possible price) obeys an inverse power law.

Proof:

$$g(\mu) = \lambda e^{-\lambda\mu} \quad \text{exponential distribution, p.d.f.}$$

$$G(\mu) = 1 - e^{-\lambda\mu} \quad \text{exponential distribution, cumulative}$$

$$H(p/p_0) = 1 - (p/p_0)^{-\lambda} \quad \text{Theorem 12, } p = p_0 e^\mu$$

$$h(p/p_0) = \lambda (p/p_0)^{-\lambda-1} \quad \text{Differentiate to get the p.d.f. } \blacksquare$$

This axiom should be uncontroversial since it does not disturb any previous results and proposes the most rudimentary distribution available; a complicated formula with many *ad hoc* features would need to be justified, but how much justification does the exponential need?

Computers are quite powerful now and, if econophysicists want to be helpful, their statistics may be used to determine the parameters for my theory. But, 14 years after Gopikrishnan, they must admit that statistics do not *create* theory. Only the axiomatic method can create new theory.

References

Aguilar, Victor. 1999. *Axiomatic Theory of Economics*. Hauppauge, NY: Nova Science Publishers, Inc.

Buchanan, Mark. 2013. *Forecast*. New York, NY: Bloomberg USA

Gopikrishnan, Meyer, Amaral and Stanley. 1998. “Inverse cubic law for the distribution of stock price variations” **European Physical Journal B**. 3 (1998) 139

Kolmogorov, A. N. [1933] 1956. *Foundations of the Theory of Probability*. Nathan Morrison trans. New York, NY: Chelsea Publishing Company