Trigonometry of Rational Numbers

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Four millennia ago it was not Greece but Babylon that was the seat of learning; by the time of Euclid, Babylon was in ruins. They knew of the square and the equilateral triangle; and they knew how to bisect an angle. By bisecting the central angles of the former once and the latter twice, it is easy to divide the circle into 24 parts. But this angle measure was not fine enough for mariners, so they divided the circle into 360 parts. Ease of construction does not explain 360 because trisecting an angle is impossible with compass and straightedge. Given the value that ancient gods put on accuracy (e.g. the ones at Delos who would sooner see plague than have sides of 1.26 when doubling a cube), 360 is not explained by the 365.25 days in a year. Nobody today really knows why there are 360 degrees in a circle and 60 minutes in a degree.

This decision – not the Hanging Gardens or Hammurabi's Code – is their legacy. Until the invention of electronic calculators in 1980, all practical applications of trigonometry required rounding angles off to the nearest minute of angle. Mathematicians understood that sine and cosine have Taylor Series expansions and are thus smooth functions that can be evaluated at any angle to any desired precision, but everybody else saw them as discrete functions that were "evaluated" by table look up. My 1981 copy of *CRC Standard Mathematical Tables* devotes 23 pages of fine print to the trigonometric functions in minutes of angle to five decimal places.

Brook Taylor first described his series in a 1717 calculus book, about two millennia after Euclid published *The Elements*. Mathematicians waited another 150 years for a rigorous definition of fields so trigonometry could include irrational numbers like $\pi, \sqrt{2}$ or $\sqrt{3}$. Dedekind wrote in his 1871 supplement to the second edition of Dirichlet's text on number theory:

> *By a field we mean any system of infinitely many real or complex numbers, which in itself is so closed and complete, that the addition, subtraction, multiplication, and division of any two numbers always produces a number of the same system.*

	Addition	Multiplication
Associative	$(a+b)+c=a+(b+c)$	$(ab)c=a(bc)$
Commutative	$a+b=b+a$	$ab=ba$
Identity	$a+0=a=0+a$	$a\cdot 1=a=1\cdot a$
Inverses	$a+(-a)= 0=(-a)+a$	$aa^{-1}=1= a^{-1}a$ if $a\neq 0$

Addition and multiplication are related by the Distributive Axiom: $a(b+c)=ab+ac$

Brahmagupta invented the additive identity zero about a millennium after Euclid. A rigorous definition for the field of rational numbers was within his grasp! It was only the existence of irrational numbers that stayed his hand. Mathematicians thought that, because π, $\sqrt{2}$ and $\sqrt{3}$ exist, they had to be included in the definition of a field. It did not occur to them that such troublesome numbers could simply be excluded from the definition of a field and, when they came up in real-life trigonometry problems, estimated with rational approximations. Apparently, if they could not be exact, they did not want to do the work at all. The existence of irrational numbers put a stop to the study of trigonometry like a chalk line stops a hen.

Trigonometry to six decimal digits of accuracy was possible in Brahmagupta's day, but it would have required that he redefine the unit of angle. Also, he would have had to come to grips with the fact that Dedekind was still a millennium in his future, so he needed to just get busy studying trigonometry within the context of the field of rational numbers. $\frac{355}{113} = 3.1415929$ is near $\pi = 3.1415927$. (All floating point numbers in this paper are reported to eight digits.)

Define the sub-degree to be one 213$^{\text{th}}$ part of a degree; the angle subtended by a .30 caliber bullet hole in a 100-yard target. Define the r-circle to be the union of 51120 congruent isosceles triangles with concurrent vertices, each with a vertex angle of one sub-degree. Their legs are exactly 8136 times longer than their bases. In an r-circle with circumradius R, the length of the perimeter is $\frac{51120}{8136}R = \frac{710}{113}R = 6.2831858R$ and, by Heron's formula, the area is $\frac{51120}{4 \cdot 8136}R\sqrt{4R^2 - \frac{R^2}{8136^2}} = \frac{355}{113}R = 3.1415929R^2$. Verify accuracy at the common angles:

$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}} \approx \frac{5753}{8136} = 0.7071042$ The correct answer is: 0.7071068

$\sin 60° = \cos 30° = \frac{\sqrt{3}}{2} \approx \frac{7046}{8136} = 0.8660275$ The correct answer is: 0.8660254

Note that sine and cosine have about a tenth the accuracy of the approximation of π and, indeed, accuracy is proportional to the size of the denominator. So why not use $2^{15} = 32768$ for the denominator? The accuracy of $\frac{102944}{32768} = 3.1416016$ for the area of the unit circle is about a hundredth the accuracy of $\frac{355}{113} = 3.1415929$, though comparison of sine and cosine evaluated at arbitrary angles indicates that these numbers in parts per 32768 average about four times their accuracy in parts per 8136. How can adding up things that are individually four times more accurate result in a sum that is a hundred times less accurate? It is because the errors all overestimate the triangles' areas and so they accumulate, while the larger errors of my method are more evenly distributed around the correct answers and thus errors cancel out.

Thus, numerical integration of functions defined in cylindrical or spherical coordinates, when performed on a 16-bit computer, is more accurate using my definition of trigonometry. For high school students, numerical integration is roughly defined as finding the surface area or volume of regions bounded by curved surfaces; that is, integral calculus is basically quadrature theory extended to the sum of innumerable but infinitesimally small regions. Furthermore, while this is far beyond high-school mathematics, all numerical solutions of differential equations are equivalent to finding the area or volume of an associated region. And solving differential equations is most of the mathematics needed by engineers.

For instance, a tiny 16-bit computer riding in the nose of an Anti-Tank Guided Weapon (ATGW) is evaluating trig functions that are at least recognizable to high school students, but doing so over and over really fast so the computer can put a feather touch on the fins every few microseconds to keep the missile on target. When it arrives, its computer has – in a way of thinking – found the area of a region by partitioning it into a large number of very small regions and adding up their areas. How accurate this summation is, which depends largely on how much the individual errors cancel each other out, is what determines accuracy of the missile.

A table of the numerators for the denominator 8136 is given on the following page, all in hexadecimal, and incremented in degrees. Programmers compiling a more precise table should increment in thirds of a degree. Like the printed tables of decimal numbers that existed before 1980, it gives only sine and cosine and only from $0°$ to $45°$. Instructions are provided for evaluating sine and cosine at angles in the range $-360°$ to $360°$. Instructions are also provided for using sine and cosine to find tangent, cotangent, secant and cosecant. The need to follow these instructions every time they evaluated trig functions meant that pre-1980 students became intimately familiar with the fact that sine and cosine are cyclic and are offset from one another, as well as how the other trig functions are defined in terms of sine and cosine.

Modern calculators just return a decimal number at the push of a button, which is convenient and accurate, but it also means that students tend to accept these numbers at face value and never actually visualize sine and cosine being cyclic and offset from one another. Standardized exams often ask questions about the cyclic nature of trig functions and the offset between sine and cosine that would have been a gimme for any pre-1980 student. Modern students stumble over such questions because – while they have seen the graph at least once – it was not something they needed to know to evaluate the trig functions in their homework assignments.

Thus, instructors who are teaching only for the standardized exams should not dismiss this paper on the grounds that those exams will not ask about the trigonometry of rational numbers, but will just provide a virtual 8-digit calculator. Practice makes perfect!

Sin		Cos/Sin		Cos
0	0.000	5A	1FC8.0	0
1	8D.FE	59	1FC6.C	1
2	11B.F	58	1FC3.1	2
3	1A9.D	57	1FBC.E	3
4	237.9	56	1FB4.3	4
5	2C5.2	55	1FA9.1	5
6	352.7	54	1F9B.7	6
7	3DF.8	53	1F8B.6	7
8	46C.5	52	1F78.D	8
9	4F8.C	51	1F63.D	9
A	584.C	50	1F4C.6	A
B	610.6	4F	1F32.8	B
C	69B.9	4E	1F16.3	C
D	726.3	4D	1EF7.8	D
E	7B0.4	4C	1ED6.5	E
F	839.C	4B	1EB2.C	F
10	8C2.9	4A	1E8C.D	10
11	94A.C	49	1E64.8	11
12	9D2.3	48	1E39.D	12
13	A58.D	47	1E0C.C	13
14	ADE.B	46	1DDD.5	14
15	B63.B	45	1DAB.A	15
16	BE7.D	44	1D77.9	16
17	C6B.0	43	1D41.4	17
18	CED.3	42	1D08.A	18
19	D6E.7	41	1CCD.C	19
1A	DEE.9	40	1C90.9	1A
1B	E6D.B	3F	1C51.4	1B
1C	EEB.A	3E	1C0F.B	1C
1D	F68.7	3D	1BCB.E	1D
1E	FE4.0	3C	1B86.0	1E
1F	105E.6	3B	1B3D.F	1F
20	10D7.7	3A	1AF3.C	20
21	114F.3	39	1AA7.7	21
22	11C5.9	38	1A59.1	22
23	123A.A	37	1A08.A	23
24	12AE.4	36	19B6.3	24
25	1320.6	35	1961.B	25
26	1391.0	34	190B.4	26
27	1400.2	33	18B2.E	27
28	146D.C	32	1858.9	28
29	14D9.B	31	17FC.5	29
2A	1544.1	30	179E.4	2A
2B	15AC.C	2F	173E.5	2B
2C	1613.C	2E	16DC.9	2C
2D	1679.0	2D	1679.0	2D

Trigonometric Functions in Hexadecimal

← Numerators for Denominator $1FC8$

For angles from $2D°$ to $5A°$, read cosine to the left and sine to the right.

For angles from $5A°$ to $B4°$, use these:

$$\sin(\theta) = \cos(\theta - 5A°)$$

$$\cos(\theta) = -\sin(\theta - 5A°)$$

For angles from $B4°$ to $168°$, use these:

$$\sin(\theta) = -\sin(\theta - B4°)$$

$$\cos(\theta) = -\cos(\theta - B4°)$$

For negative angles, use these:

$$\sin(\theta) = -\sin(-\theta)$$

$$\cos(\theta) = \cos(-\theta)$$

For tangent [cotangent], divide the numerator of sine [cosine] by the numerator of cosine [sine].

For secant [cosecant], divide $1FC8$ by the numerator for cosine [sine].

Verify these identities with some angles:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\sin(\theta)}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos\theta} \qquad \cot\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1-\cos(\theta)}$$