Towards Exact Solutions Of The American Style Options

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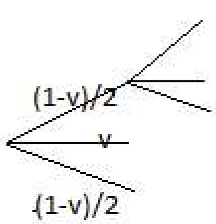
Abstract:
Exact Solutions of the American style call or put option on the same footing as the European (Vanilla) call or put option have not been derived. That is a function that describes the call option say as depending on the underlying & portfolio hedge & portfolio time dependent riskless asset increase in value & yet inclusive of the uncertainty due to early exercise. In this letter we present several approaches towards deriving such a function that are conceptually inspired by and are based on current methods of discrete probabilistic decision trees and their diffusion approximation, & from methods of applying boundary conditions that are here additionally random in time. We derive under a certain set of limiting assumptions re the random in time boundary conditions & their statistics (utilizing for illustration the simplest) and furthermore for flux allowing VN boundaries & flux absorbing Dirichlet boundaries a closed form solution.

Introduction:
The American style option is a European style option with further uncertainty due to the possibility of early & random as in unknown in time exercise. As such it is European style option between such increments of time when decisions (discrete) are made as to exercise or to hold. In equation form [1,2] this is

$$\frac{\partial}{\partial t} f(x,t) - r(1 - x\frac{\partial}{\partial x}) f(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} f \le 0 \quad (1)$$

where the usual portfolio construction of a European style call option is used to derive the LHS Black-Scholes style PDE yet this is no longer equated with zero.
An alternative discrete form is from Hull et al. [3] & gives 'a' trinomial form with Pup, Pdown, & Pr (risk-free) with P=1-Pu-Pd, and with a symmetrical example of this for illustration:



where as one realization at t1 with increments of decision time as t1,t2,..., the probability not to exercise is v , the remaining decisions such as sell 1-v the binomial model & if as here another decision is to buy then buy=(1-v)/2 & sell=(1-v)/2 if symmetrical and with various other partitioning possible including random time increments.

Note that even in this model the assumption is that between decisions the evolution is that of a European style option as by Black-Scholes that is Eq.1 above but with the inequality replaced with an equality to zero.

Some authors have shown the discrete model in some limit is that of a Finite element PDE solution of the Black-Scholes equation under various boundary conditions assumptions [3].

These several models then are our starting points towards deriving functions of a closed form nature for the American style early exercise possibility options.

Introduction II:

Much can be done to obtain perhaps more simple & useful functions that describe the American style options. For example a binomial or trinomial probabilistic decision branching process is in the continuous limit where useful Black-Scholes style functions for European style options are found, is a continuous Fokker-Planck or Kolmogorov PDE partial differential equation. We shall examine this in light of the boundaries & of options.

Another approach possible to us is that of boundary conditions of PDEs...the decision at some increment with the decision random & perhaps the increment random as well is then a boundary condition. It seems daunting that the boundary condition is random but this should not be so, as a boundary condition BC is a mathematical device & random things are a mathematical device and we know how to deal with both mathematically and such that the mathematical devices then perhaps model well the observed phenomenon. As such we will solve the random boundaries problem of an otherwise Black-Scholes type of backwards Fokker-Planck PDE equation.

Model I, that of discrete decisions with probabilities:

A trinomial model is, with discrete deterministic increments of time as here, a modified binomial model and it resembles for many such decisions a master equation formula [4]

$$\frac{\partial}{\partial t}P(x,t) = \int [(w(x,x')P(x,t\,|\,x',t') - w(x',x)P(x',t'\,|\,x,t)) - w(x,x)P(x,t\,|\,x',t')]dx' \qquad (2)$$

Expanding the RHS in a power series [4] about x=x'+ct results in a (infinite) moments partial differential equation as in C.W.Gardiner's handbook of stochastic methods (other methods are parameter & system size Van Kampen expansions) [4]. For example most simply and as is easily recognized for the binomial model

$$\frac{\partial}{\partial t}P(x,t) = \sum_{n=1}^{\infty}(-1)^n \frac{\partial^n}{\partial x^n}[M_n(x,t)P(x,t)]$$

and it is customary for Gaussian based extensive statistics to cut off the series at M2 that is the first (conditional) moment corresponding to the drift coefficient and the second conditional moment corresponding to the diffusion coefficient are included [4] while if one chooses to include higher moments it is possible to do so but at the cost of transforms that are self-referring rendering the resulting PDE inherently nonlinear in probability density functions [5]. That is, a third order moment is or can be, as iterratively higher moments, recursive folded into a second order moment as follows [5]

$$\frac{\partial^2}{\partial x^2}[c_2(x,t)P] => \frac{\partial^3}{\partial x^3}[C_3 P]$$
$$c_2 = C_3 \nabla \ln C_3 P$$

where this can be performed to increase or decrease order of partial differentiation as well as a general method of transforms...the cost is that the moments now depend on the probability itself rendering the equation nonlinear explicitly in density functions.

The specific moments are if 1st drift & second diffusion those of the discrete branching process, & these are for the forward branching process the standard deviation & variance of the branching processes respectively.

And as an ad hoc first guess, we have a forward Black-Scholes two-point or transition probability PDE in addition or superposition L1+L2 of operators, where with this continuous approximation of the discrete decision branching process (bionomial or trinomial decisions, discrete time increments or additionally random time increments) and where it is stated that this composition or superposition of operators is a (total) continuous approximation to the discrete or total evolution, both the discrete decision 'jumps' & the continuous B-S between jumps.
Also we want to point out that the inequality of Eq.1 is not supposedly satisfied but the equation can be made into an equality with zero as

$$\frac{\partial}{\partial t} f(x,t) - r(1 - x\frac{\partial}{\partial x}) f(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} f + L_2(x,t) = 0$$
$$L_2 \geq 0 \qquad (4)$$

where the deficit function L2, an operator(s) acting on a function, in detailed view changes the form of the inequality identically such that an equality is satisfied, is greater than or equal to zero itself, and has the interpretation that it is the uncertainty associated with the early exercise, & this generally the view both from the trinomial (or binomial) discrete process which was our 'conceptual' starting point, & is also the composition as by superposition of operators that here correspond to the first orde drift & the second order moment diffusion or rather uncertainty & in general by extension of the process of superposition can be made at the function(al) or operator levels.

And it should be noted that equation Eq.4 is onto itself a starting point for tries for derivation of an American Black-Scholes equation, as the Eq.1 obstacle is the inequality (otherwise a solution is obtained as European Black-Scholes), & 'forcing' an equality as by analogy to NLP nonlinear programming whereas deficit & surplus functions are utilized to transform inequalities to equalities is being applied here, from one view point. As an example one can require that a) L2 is a function b) L2 is an operator(s) on a function(s).

And for completeness at least of the economics that motivate this mathematics, what does the L2 correspond to in terms of real hard assets & equities & hedges & riskless investments...meaning that given a portfolio

$$\Pi(t) = \Pi_o e^{rt} \le f - x\Delta$$

$$d\Pi \le df - dx\Delta$$

$$d\Pi \overset{?}{=} df - dx\Delta + F[L2]$$

requiring that at all times the equality is re-instated as by a hedge (the delta in the above portfolio) or financial instrument or asset or equity is very much a question (as typed) of economics and must be addressed as such. An alternative is self-referring deficit or surplus modifiers being the L2 equalizing function (i.e.. functionals F(f)), but this too has an economic interpretation...one of discounts or premiums paid out or taken in. We return to these questions in future work.

Model II, that of random boundary conditions:

The previous model I had as discussed a continuous process of the Black-Scholes type that then in precise intervals (it is a model assumption... random intervals as generalizations are possible) would undergo a decision making process which depended probabilistically on its outcome...either a) the option was exercised & became zero valued b) was held onto with no change in value c) was added to increasing its value.

The point of decision is strictly speaking a boundary...the continuous evolution as by a Fokker-Planck 'diffusion' of the forward Black-Scholes type, looking at this from t1 to t2 to t3 etc..., is interrupted by the boundary.

a) If at the boundary the decision was to exercise or sell off, the value goes to zero...this is an absorbing boundary condition, a Dirichlet type of boundary condition.

b) If held onto with no change in value, then this re-propagates forward with no change & we can say this is a 'current or flux through allowing' boundary & of the Von Neumann boundary condition type.

c) the 'third' decision is not really a boundary condition, that of buying additionally or adding value is actually merely an added value or premium & is discussed elsewhere & is merely the underlying paying out a dividend in reverse & will not be discussed here further as there is much literature on the subject. In summary this too is a propagating across a discontinuity now nonzero valued, or a VN type as (b).

There is one final wrinkle to this problem before we start deriving. in B.C. boundary conditions the boundaries are in actuality the (vector of..) variable(s)... herein corresponding to price...i.e.. price, heat, energy, position, coordinate are usually encountered vector variables & observables...not however a boundary that of time. This seems to have contributed much to the difficulty in deriving a solution.

Therefore we will perform a trick...we will transform the PDE of f(x,t|x',t') the forward Fokker-Planck dependent on 2nd order on price x,x' & to first order on time t,t' to a PDE dependent to 2nd order on time t,t' & to first order on price x,' & this of f(x,t|x',t') ...noting that mathematicians would insist on a separate notation characterizing conditional dependence precisely, we merely retain the two-point transition density notation (...|...) which is here similar in both PDE forms.

$$\frac{\partial}{\partial t}g(x,t\,|\,x',t')+\frac{\partial}{\partial x}[a(x,t)g]-\frac{1}{2}\frac{\partial^2}{\partial x^2}[b^2(x,t)g]=\delta(x-x')\delta(t-t') \tag{5}$$

This the 'free' propagation, it is written as the delta functions source Green's function g(x,t|x',t') , & noting that f(x,t) is the one point function of a source less Eq.5 and is that of a forward Black-Scholes PDE & a,b then are the drift & diffusion coefficients of that B-S equation.

We now use that order or power of partial differentiation transform we noted in the model I aside... The transformations are

$$\frac{\partial}{\partial t}[(1+h)g]-\frac{\partial}{\partial t}[hg]$$

$$\frac{\partial}{\partial t}[(1+h)g]=>$$

$$(1+h)=D\nabla_t \ln g$$

while the reducing by an order transformations of the price evolution operators are

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}[b^2(x,t)g]+\frac{\partial}{\partial x}[k(x,t)g]-\frac{\partial}{\partial x}[k(x,t)g]$$

$$k(x,t)=\frac{1}{2}b^2\nabla_x \ln b^2 g$$

and putting these two together remainders after some algebra of gathering of terms & then defining gamma,

$$\frac{D}{2}\frac{\partial^2}{\partial t^2}[\gamma/(k-a)]-\frac{\partial}{\partial t}[h\gamma/(k-a)]-\frac{\partial}{\partial x}[\gamma]=\delta_t\delta_x \tag{6}$$

here deltas are as before but in abbreviated notation, & gamma is

$$(k-a)g=\gamma$$

& a,b are as before the drift & diffusion of the European Black-Scholes option.

We now move to boundary conditions & matching rigorously the same as by Green's theorem...while we could forego all of this in favor of solving time dependent B.C., for clarity & as to how to handle randomness in such B.C. in time here 'promoted' to coordinate, we proceed as we are by applying Green's theorem to a parabolic 2nd order diffusion Fokker-Planck equation in source Green's function form.

We transform the 1st order 'parameter' of price to a 'momentum' like constant as by a Fourier transform

$$\gamma(t-t',x-x')=\int e^{iq(x-x')}\gamma(t-t',q)dq$$

this resulting in the PDE

$$\frac{D}{2}\frac{\partial^2}{\partial t^2}[\gamma/(k-a)]-\frac{\partial}{\partial t}[h\gamma/(k-a)]-iq=\delta_t \tag{7}$$

Boundary conditions:
Applying boundary conditions consists of defining an 'overall' Green's function which spans the entire period of time of the duration of the option [6], inclusive of the 'free' inter decision propagation & beyond to the decisions periods & then the other free propagations (if yet to be held)...

$$\begin{aligned}
&\Gamma(t\,|\,t')=\gamma(t-t')_\alpha\Theta(t,t',\alpha)+\int_{t_o}^{t_f}\gamma(t-t'')_\alpha\Theta(t,t'',\alpha)V_\tau(t'')\Gamma(t''\,|\,t')dt''\\
&\alpha=\tau_<,\tau_>\\
&\Theta(t,t',\tau_>)=\theta(t-\tau_>)\theta(t'-\tau_>)\\
&\Theta(t,t',\tau_<)=\theta(\tau_<-t)\theta(\tau_<-t')
\end{aligned} \tag{8}$$

This full Green's function is therefore comprised of the pre decision & post decision, 'tau' the time of decision (boundary) a random number however assumed mathematically as constant for now.
This integral form can be written in operator form as

$$\begin{aligned}
&L_\Gamma=\theta(\tau_<-t)L_<+\theta(t-\tau_>)L_>\\
&V_\tau(t)=[\delta(t-\tau_<),\frac{\partial}{\partial t}]_+ +[\delta(t-\tau_>),\frac{\partial}{\partial t}]_+\\
&(L_\Gamma+iq)\Gamma(t\,|\,t')=\delta(t-t')
\end{aligned} \tag{9}$$

where the '+' denotes anti-commutation. Some readers will recognize this resulting integral or differential boundary coupling as that obtained by T.E.Feuchtwang by Green's theorem, by Datta et al. in 'atomistic' discrete matrix elements & by Michael and Johnson by the continuous self-energy of coupling method [6].
The solution of these equations Eq.8,9 is the full Green's function

$$\begin{aligned}
&\Gamma(t\,|\,t')=\gamma(t-t')_<\theta(\tau_<-t')+\frac{D}{2}[\partial_{t''}\gamma(t-t'')_<.\Gamma(t''\,|\,t')-\gamma(t-t'')_<.\partial_{t''}\Gamma(t''\,|\,t')]_{t''=\tau_<}\\
&(t<\tau_<)
\end{aligned} \tag{10}$$

and similarly for the after decision portion.

Random boundary:
The random nature of this decision enactment & its effects enters into the above mathematics as in the expression for boundary matching, that is the Dirac delta function(s), which can be written as a Fourier

integral to facilitate averaging over the random values as

$$\delta(t-\tau_<) = \frac{1}{2\pi}\int e^{is.(t-\tau_<)} ds$$

$$<< \delta(t-\tau_<) >>= \frac{1}{2\pi} << \int [1 + is.(t-\tau_<) + \frac{(is.(t-\tau_<))^2}{2!} + ...] ds >> \quad (11)$$

where <<..>> denotes averaging over the random variable tau, itself from the exponential expansion an infinite series of powers of.

Alternatively only the random variable can be averaged as

$$<< \delta(t-\tau_<) >>= \frac{1}{2\pi} << \int e^{ist} [1 + (-is\tau_<) + \frac{(-is\tau_<)^2}{2!} + ...] ds >>$$

$$<< \tau_<^2 >>= 2D(x - x_o)$$

$$<< \tau_< >>= 0$$

$$<< [1+....] >>= \frac{1}{2\pi}\int e^{ist}[1 + 0 - \frac{s^2 << \tau_<^2 >>}{2!} + 0 + \frac{s^4 << \tau_<^4 >>}{4!} + - higher] ds >>$$

$$<< \tau_<^4 >>= [<< 12 >><< 34 >> + << 13 >><< 24 >> + << 14 >><< 23 >>]$$

$$= \frac{1}{2}\binom{4}{2}(<< \tau_<^2 >>)^2 \quad (12)$$

Firstly it will be noted that all odd powers of tau will vanish (equal zero) as we assume the random increment is zero mean. Secondly the even powers are pair-wise to be averaged to non-zero valued variances as they have the form

$$<< v^2 >>= \frac{\int v^2 K(v) dv}{Z} = \frac{\int v^2 e^{\frac{-v^2}{\sigma^2}} dv}{Z} \quad (12)$$

for the variance, but higher order decompositions of co variances of variances (pair-wise like decomposition) is possible. Other statistics in Eq.12 can be utilized, for eample one can utilize the nonextensive or power law statistics these well adapted to history dependent statistical modeling. For Gaussian uncorrelated however our simple assumption (for illustration & to show the simple results), we have the simplest decomposition that of pair-wise. Moreover here the random variable is merely powers of the same variable & we have a problem of evaluating moments.

$$\frac{(is.(t-\tau_<))^2}{2!} = \frac{1}{2!}[(-1)(t^2 - 2t\tau_< + \tau_<^2)]$$

$$<< \tau_<^2 >>= 2Dx \quad (13)$$

where we merely evaluate the averaging of the random variable now. We then obtain the desired results by the replacement of (13) in (9) & so on...The result is the replacement for Gaussian randomness (2nd moment evaluated) at the boundary(s)

$$V_\tau(t) = [\delta(t - \sqrt{2Dx_<}), \frac{\partial}{\partial t}]_+ + [\delta(t - \sqrt{2Dx_<}), \frac{\partial}{\partial t}]_+ \qquad (14)$$

dimensional analysis is simply: D goes as seconds^2/dollar & thus the argument of the diffusion goes as seconds or generally here as time.

Simplifications, Assumptions, Approximations:
With the (14) replacement, we rederive the operator & Green's function (overall) to obtain a result that includes the full boundary conditions...however for illustrative purposes we simplify & assume we are interested in propagating solutions as VN and obtain

$$\Gamma(t \mid t') = \gamma(t - t')_< \theta(\tau_< - t') + \frac{D}{2}\partial_{t''}\gamma(t - t'')_< . \Gamma(t'' \mid t')\big|_{t''=\sqrt{2Dx_<}}$$

$$(t < \tau_< \sim \sqrt{2Dx_<})$$

$$(L_\Gamma + iq)\Gamma(t \mid t') = \delta(t - t') \qquad (15)$$

We therefore have the full solution (a solution rather) of VN boundaries across uncertain BCs & its full operator Green's function equation.

Re transformation to 2nd order price & first order time:
The transformation mathematical trick shown to be useful, we should re transform to the traditional picture...noting however that (15) is actually an exact solution given the assumptions & could be utilized to measure value & price as functions of time, it is merely not in the usual form.

From (15, 7, 6) we should have

$$(L_\Gamma + \nabla_x)\Gamma(x, t \mid x', t') = \delta(t - t')\delta(x - x')$$

$$G(x, t \mid x', t') = F(\Gamma(x, t \mid x', t'))$$

(16)

where inverse Fourier transforms to x,x' & operators of those 'parameters' are rewritten in order to re transform to 2nd order in x,x' & first order in time parameter...
However G(..|..) now is the full American option written as 2nd order x,x', & yet has the uncertainty due to the decisions of early exercise from the (15) results. We can carry out the inverse transform however as we have obtained one exact result given our assumptions which as pointed out has the price, uncertainty due to early exercise & time parameterization, it suffices for our derivation in this first letter. We shall return to the G(..|..) in following reports.

Conclusion:
We have discussed some of the difficulties in obtaining closed form solutions of American style options with early exercise risk/reward uncertainty.
We discuss the trees models, branching/decision & their results in the literature. We utilize this discussion to hem in the risk/reward uncertainty in early exercise. We thus obtain what the form the portfolio derivation must have in order to reach certainty in mathematical description i.e., taking into account the uncertainty exactly.
We then present a novel form of an exact closed form solution under certain assumptions of boundary conditions & standard form of statistical uncertainty in continuous approximation. This allows us to derive a full evolution pricing formula of an American style option, that is an instrument of a portfolio

that takes early exercise uncertainty exactly albeit simplistically. It is obvious from our derivation that the statistics of exercise need not be Gaussian white noise correlated but all the popular statistics, from pink to brown to history following & correlated (say nonetensive, Tsallis or Beck type) can be applied. As a robust statistics is that of the nonextensive statistics of Tsallis, it is immediately obvious that the moment replaced by ~$P(x,t)^{(1-q)/2}$ with P(x,t) the Tsallis power law function/PDF here the conditional 2nd moment result, & this history following conditional moment or diffusion coefficient result is of obvious utility as -parameter power law value from the underlying variable's price statistics enters directly into the analysis with imparting of high accuracy in quantifying uncertainty as previously reported by us & others.

Additionally the initial form of the 'freely propagating' European B-S is here the log normal (i.e.. Gaussian like) underlying...we have in previous letters generalized the European freely propagating B-S to the derivative pricing with nonextensive (Tsallis) statistics [7]. Therefore in two separate instances, choice of statistics & therefore accuracy enter as assumptions of our models, these not otherwise impacting the import of this letter, that early exercise uncertainty can be quantified exactly with the caveat that the form of the uncertainty is model-of-statistics specific.

In the future we will exactly derive the trees continuous approximate PDEs , the surplus/deficit function of the parallel compensated of risk American portfolio, & therefore compare our results with the BC boundary conditions derived exact results (closed form) reported in this letter.

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