Article

Spurious memory in non-equilibrium stochastic models of imitative behavior

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Abstract: The origin of the long-range memory in the non-equilibrium systems is still an open problem as the phenomenon is reproduced using Markov processes. In these cases a notion of spurious memory is introduced. A good example of Markov processes with spurious memory is stochastic process driven by non-linear stochastic differential equations (SDE). This example is at odds with models built using fractional Brownian motion (fBm). We analyze differences between these two cases seeking to establish possible empirical tests of the origin of the observed long-range memory. We investigate probability density functions (PDF) of burst and inter-burst duration in numerically obtained, while solving non-linear SDE, time series and compare with the results of fBm. Our analysis confirms that the characteristic feature of the processes described by the one-dimensional stochastic differential equations is the power-law exponent $3/2$ of the burst or inter-burst duration PDF. This property of stochastic processes might me used to detect spurious memory in various non-equilibrium systems, where observed macroscopic behavior can be derived from the imitative interactions of agents.

Keywords: spurious memory; non-equilibrium systems; agent-based and stochastic modelling; Markov processes; first passage times

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1. Introduction

The application of statistical physics to divers fields such as social sciences and economics, biology and population genetics, medicine, information technology, computer science, etc. [1?] is making this interdisciplinary research very universal. Though the number of agents is usually incomparable with number of particles in physical systems, the understanding of macroscopic behavior of social and biological systems naturally invokes methods of statistical physics with very simplified interactions of individuals. Humans and biological entities are itself complex systems with unknown detailed behavior and any attempt to reproduce microscopic interactions of agents might look unrealistic. Thus the probabilistic description of agent interactions in social systems looks the most appropriate and natural. Even in very simplified models of agent interactions the collective behavior may lead to the ordered or disordered states. How the disordered interactions of agents create order in macroscopic behavior is very interesting question, nevertheless, the cases when non-equilibrium fluctuations do not disappear in the system are of high importance as well.

There is a limited number of solvable, mathematically transparent many-body systems and Ising model with its Glauber dynamics is among the most fundamental examples [2]. Being a very popular tool for the investigation of transition from order and disorder states, Ising model is solvable only in one-dimensional case and local interactions of spins [?]. Nevertheless, Ising model is useful to the modeling of opinion and population dynamics and gives motivation to many other applications of statistical mechanics [?]. It is possible to simplify pairwise interactions of agents in the way, which leads to the solvable cases of many body systems in any dimension. The voter model is a good example of such social system widely used in modeling of opinion dynamics and population genetics [3–5]. In one-dimensional case the voter model coincides with one-dimensional Glauber dynamics and can be considered in other dimensions and various topologies of agent interactions. The standard voter model converges to the consensus of opinions and this is related with two circumstances: local nature of interactions and imitation of neighbor opinion without idiosyncratic decision making. From our point of view, the case of global agent interactions or system running on the randomly generated network, including idiosyncratic switching of opinions, is of great importance as exhibits continuing stochastic fluctuations in collective behavior [?]. The evolution of such system can be written as the Focker-Planck equation or as a non-linear SDE for population evolutions and can be seen as a special case of voter model [6,7], of Moran model [1] or of Kirman's model [8,9]. The continuing fluctuations in such a non-equilibrium system with imitative behavior of agents exhibit very general power-law scaling properties including spurious memory applicable to social [10], financial [11–13] or biological [1] systems.

Here we investigate the long-range memory property, which might originate from the true long-range memory process with correlated increments such as the fractional Brownian motion (fBm) [14–16] or from the stochastic processes with non-stationary uncorrelated increments [15–18]. There is a fundamental problem to find out which of the possible alternatives, fBm or diffusive processes with non-stationary increments, is in the origin of observed long-range memory. Here we employ the dependence of first passage time PDF on Hurst parameter H for the fBm [19,20] and apparently different behavior for non-linear diffusive processes [21]. This explains that the long-range memory present in social, financial and biological systems can arise from non-linear agent interactions and non-linear transformations of the population time series.

In Section **??** we present the short theoretical background for our empirical investigation, in Section **??** we deal with empirical data from Forex and in Section **??** we discuss and conclude results.

2. Non-equilibrium stochastic fluctuations arising from the imitative behavior of agents

One agent (particle) jump Markov processes have become an efficient tool in modeling of physical, biological and social systems [1,10,22]. The microscopic behavior of agent is replaced by continuous time Markov processes with specified transition rates. In the system with large number of agents N and two choices of opinion or state, for example (0,1), there are two possible one step (birth-death) changes of system macroscopic state: a) the number of agents n in state 1 increases or b) decreases. Such a simple but general enough definition of opinion or population dynamics can be specified by two transition rates

$$p(n \to n+1) = (N-n)\mu_1(n,N) \equiv p^+(n,N), \tag{1}$$

$$p(n \to n-1) = n\mu_2(n,N) \equiv p^-(n,N). \tag{2}$$

These rates define the master equation for PDF of macroscopic state evolution $P(n,t)$

$$\frac{\partial P(n,t)}{\partial t} = p^+(n-1,N)P(n-1,t) + p^-(n+1,N)P(n+1,t) - (p^+(n,N) + p^-(n,N))P(n,t). \tag{3}$$

In the limit of high N values one can introduce normalized variable $x = \frac{n}{N}$ and write Fokker-Planck equation for PDF evolution

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[x\mu_2 - (1-x)\mu_1\right]P(x,t) + \frac{1}{2N}\frac{\partial^2}{\partial x^2}\left[(1-x)\mu_1 + x\mu_2\right]P(x,t), \tag{4}$$

which can be associated with Ito stochastic differential equation for x

$$\mathrm{d}x = ((1-x)\mu_1 - x\mu_2)\,\mathrm{d}t + \sqrt{\frac{(1-x)\mu_1 + x\mu_2}{N}}\mathrm{d}W, \tag{5}$$

where W is the Wiener noise. One gets very well known result Kirman's model with global coupling of agents or interaction defined on the random Erdos-Renyi network, when rates are defined as follows [8**?**]

$$\mu_1(n, N) = \sigma_1 + Nhx, \tag{6}$$
$$\mu_2(n, N) = \sigma_2 + Nh(x-1). \tag{7}$$

The same equations of macroscopic modeling can be attributed to the voter model with idiosyncratic agent transitions and global interactions [**?**]. Kirman's model with global coupling of agents leads to the very general case in opinion and population dynamics retaining dynamic and stochastic parts of evolution in the limit $N \to \infty$. This is ensured by the same form of herding (imitation) term in both transition rates $Nhx(1-x)$ and its linear dependence on N. The relaxation term in SDE does not depend on herding and idiosyncratic terms vanish in diffusion part of SDE. Model can be generalized introducing non-linear dependence of time scale on macroscopic variable x, as was proposed in the modeling of finance to account variable trading activity [23]. Such additional non-linearity of the system probably is common feature of the real world. From our point of view this might be considered as a source of spurious memory, which has to be identified from the empirical data of real social, biological or physical systems. Lets generalize Kirman's transition rates retaining some mathematical symmetry and adding some additional non-linearity quantified by parameter α

$$\mu_1(n, N) = (\sigma_1 + Nhx)x^{-\alpha}(1-x)^{-\alpha}, \tag{8}$$
$$\mu_2(n, N) = (\sigma_2 + Nh(x-1))x^{-\alpha}(1-x)^{-\alpha}. \tag{9}$$

Certainly, there are other choices how to introduce additional non-linearity into transition rates, but this one is very well suited for our needs in this contribution. This generalization assumes that activity of agents to interact can depend on the macroscopic state x of the whole system.

As we can see from Eqs. (2-9) agent-based models with non-linear interactions can lead to the macroscopic description by a non-linear SDEs, which represent the class of Markov processes. After some additional transformation of the signal such Markov series can be represented by following or a similar form[24,25]

$$\mathrm{d}x = \left(\eta - \frac{\lambda}{2}\right) x^{2\eta-1}\mathrm{d}t + x^{\eta}\mathrm{d}W. \tag{10}$$

This Eq. has just two parameters: η as exponent of noise multiplicativity and λ as exponent of power-law PDF. In many publications it was justified that this class of SDE generates the time series with the power-law behavior of stationary PDF and PSD [25],

$$p(x) \sim x^{-\lambda}, \qquad S(f) \sim \frac{1}{f^{\beta}}, \quad \beta = 1 + \frac{\lambda - 3}{2\eta - 2} = 2H + 1. \tag{11}$$

It was demonstrated by many authors [] that Kirman's model is suitable to model return in financial markets defined as proportional to $y = \frac{x}{1-x}$. This is the reason we are more interested in series generated by such transformation of agent population. Though other non-linear transformations of x leading to very big fluctuations of the signal might be responsible for the appearance of spurious memory as well. We can write SDE for y, when x is generated by SDE (5) with rates η defined by Eq. (9)

$$\mathrm{d}y = \left(\varepsilon_1 y^{-\alpha} + (2-\varepsilon_2)y^{1-\alpha}\right)(y+1)^{2\alpha+1}\mathrm{d}t + \sqrt{2y^{1-\alpha}}(y+1)^{\alpha+1}\mathrm{d}W, \tag{12}$$

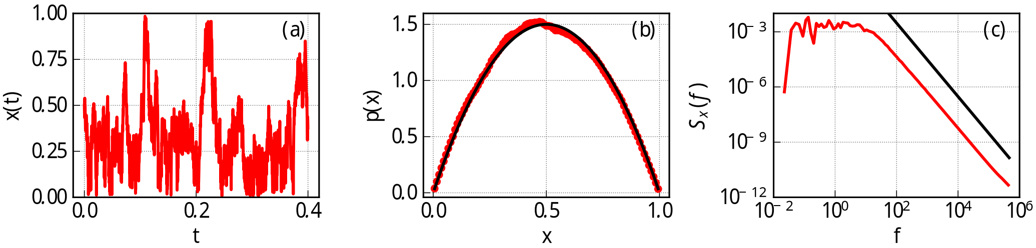

Figure 1. Fragment of time series (a), obtained by solving SDE (5) with μ given by Eq. (9), (b) PDF and (c) PSD of the same time series (red curves). Black curves in (b) and (c) represent theoretical fits: (b) $\mathcal{B}e(\varepsilon_1, \varepsilon_2)$ PDF and (c) $1/f^2$ trend line. Parameter set used in numerical simulation: $\alpha = 2$, $\varepsilon_1 = \varepsilon_2 = 0$.

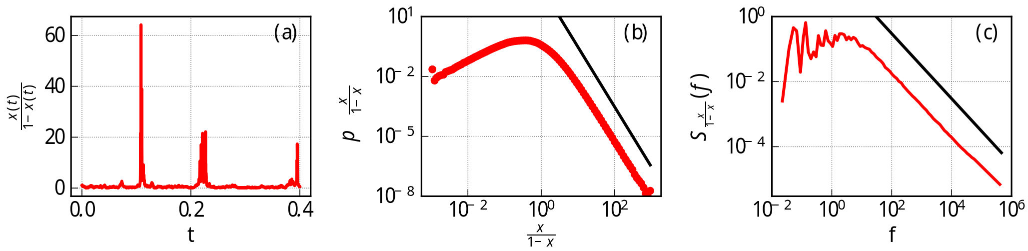

Figure 2. Fragment of the transformed time series (a), same as in Fig. 1, (b) PDF and (c) PSD of the same transformed time series (red curves). Black curves in (b) and (c) represent theoretical fits: (b) $\left(\frac{x}{1-x}\right)^{-3}$ trend line and (c) $1/f$ trend line.

Note that we scaled time here replacing $\varepsilon_1 = \sigma_1/h$ and $\varepsilon_2 = \sigma_2/h$. Eq. (12) has a unique symmetry regarding definition of y or its transformation $y \to \frac{1}{y}$. This is important as we seek to retain the symmetry in burst and inter-burst duration statistics. It is worth to note, that Eq. (12) for high values of variable $y \gg 1$ belongs to class of non-linear SDEs (10) and parameters are related as follows

$$\eta = \frac{3+\alpha}{2}, \tag{13}$$

$$\lambda = 2(\eta + \varepsilon_2 - 2) = \varepsilon_2 + \alpha + 1. \tag{14}$$

Given description of imitative behavior in agent systems can prove that non-linear functions of agent population exhibit power-law statistics, including long-range memory, which we consider as a case of spurious memory. Derived power-law properties can be confirmed by numerical calculations, see elsewhere about methods we use to solve numerically non-linear SDEs []. In Fig. 1 we demonstrate excerpt of the imitative behavior signal $x(t)$ (a), its stationary PDF (b) and PSD (c). Note that diffusion here is restricted in the region $0 < x < 1$ and PSD is $S_x \sim 1/f^2$. Only after non-linear transformation of the stochastic signal $y = \frac{x}{1-x}$, see Fig. 2, PSD becomes $1/f$ like and stationary PDF has power-law tail as given by Eq. (14). From our point of view such spurious long-range memory might originate in many social systems with imitative behavior of agents and continuing stochastic agent population or opinion dynamics. First of all such approach has to be considered as an explanation of observed long-range memory in finance []. In the next section we investigate the PDF of first passage times seeking to demonstrate that such long-range memory property is different from the case of fBm.

3. PDFs of burst and inter-burst duration in stochastic model of imitative behavior

The class of SDEs (10) describes multifractal stochastic processes [26] with non-stationary increments [15,16], power-law auto-correlation and PSD (11) and unlike for processes with correlated

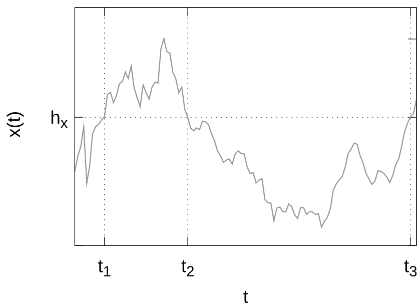

Figure 3. Exemplary fragment of a generic time series. Three threshold, h_x, passage events, t_i, are shown. Thus burst duration T can be defined as $T = t_2 - t_1$ and inter-burst duration θ can be defined as $\theta = t_3 - t_2$.

increments such as fBm, this class can be considered as having spurious memory. Here we will demonstrate clear distinction between these two different models with correlated (fBm) and uncorrelated (NSDE) increments. The idea of such distinction is based on the PDF of first return times of stochastic processes $x(t)$ with absorbing boundary at some threshold level $x = h$. Ding and Yang have considered the problem for fBm as a first return time [19].

One can define two distinct threshold passage events – one describes return to the threshold from above, while the other describes return to the threshold from below. We consider burst duration as the amount of time series spend above the threshold (starts with passage from below and ends with passage from above; $T = t_2 - t_1$ in Fig. 3). While the inter-burst duration can be introduced as the amount of time series spent below the threshold (starts with passage from above and ends with passage from below; $\theta = t_3 - t_2$ in Fig. 3). In the time series of fBm PDFs of burst and inter-burst duration coincide and can be written as [19,20]

$$p(T) \sim T^{H-2}. \tag{15}$$

The Hurst parameter H, defining the exponent of power-law PDF $H - 2$, coincides with the corresponding exponent for other one-dimensional Markov processes only when $H = \frac{1}{2}$ [5,27–29]. It was demonstrated in [21] that burst and inter-burst duration can be defined through the first passage problem with initial value x_0 infinitesimally near the threshold h. There the burst duration for the non-linear SDEs (10) was considered. The asymptotic behavior of time T PDF can be written in rather transparent form

$$p_{h_x}^{(\nu)}(T) \sim T^{-3/2}, \quad \text{for} \quad 0 < T \ll \frac{2}{(\eta-1)^2 h_x^{2(\eta-1)} j_{\nu,1}^2}, \tag{16}$$

$$p_{h_x}^{(\nu)}(T) \sim \frac{1}{T} \exp\left(-\frac{(\eta-1)^2 h_x^{2(\eta-1)} j_{\nu,1}^2 T}{2}\right), \quad \text{for} \quad T \gg \frac{2}{(\eta-1)^2 h_x^{2(\eta-1)} j_{\nu,1}^2}. \tag{17}$$

Here, $\nu = \frac{\lambda-2\nu+1}{2(\eta-1)}$, and $j_{\nu,1}$ is a first zero of a Bessel function of the first kind. The power-law behavior with exponent $3/2$ in Eq. (16) is consistent with the general theory of the first passage times in one-dimensional stochastic processes [5,28].

It is obvious from Eqs (11) that power-law statistical properties such as PDF or PSD for time series generated by SDE directly depend on non-linear transformations of the series. On the other hand these series retain some invariant property defined as PDF of first passage times T and Θ. This may serve for us as an criteria how to identify from empirical data which model of time series is better suited to describe real system having observed long-range memory properties, as any deviation of the PDF exponent from $3/2$ other than cutoff in the region of long duration has to signal that real

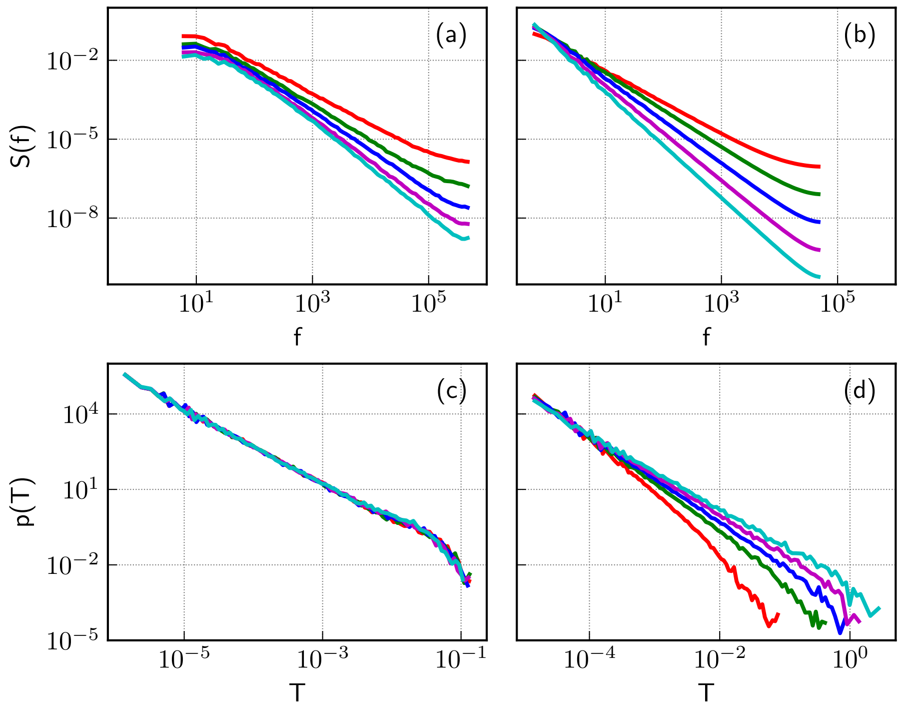

Figure 4. Comparison of PSD ((a) and (b)) and burst duration PDF ((c) and (d)) of time series generated by numerically solving Eq. (12) ((a) and (c)) with the ones obtained from fBm with relaxation ((b) and (d)). fBm parameter sets: $\gamma = 2$ (all cases), $H = 0.1$ (red curve), 0.2 (green curve), 0.3 (blue curve), 0.4 (magenta curve), 0.5 (cyan curve). Eq. (12) parameters: $\alpha = 2$, $\varepsilon = 0.6$ (red curve), 1.2 (green curve), 1.8 (blue curve), 2.4 (magenta curve), 3 (cyan curve)

long-range memory is present. Thus we investigate here statistical properties of time series generated by non-linear transformations of SDE (5) with transition rates Eq. (9), giving SDE (12). We demonstrate in Fig 4 this clear distinction of fBM and SDE by numerical comparison of PSD and PDF of T. Thus the statistical analyses of burst and inter-burst duration in empirical time series has to reveal whether signal contains real long-range memory property or such spurious property originates from non-linear memory less stochastic processes.

It is worth to define more precisely statistical properties of burst and inter-burst duration in described agent system with imitative behavior as potentially recoverable in real social systems [13]. In Fig. 5 we present numerical calculations of PDF for burst T and inter-burst θ durations with variable values of the threshold h_x. Our numerical calculations confirm the symmetry of the model for the statistics of burst and inter-burst durations regarding values of threshold h_x on both sides of mid-point $h_x = 0.5$, where PDFs of T and θ coincide. Note that fundamental power-law $3/2$ is retained for all values of threshold but burst and inter-burst durations have different positions of power-law cut off. More precise analyses confirms that cut off of PDF for burst duration T is well described by previously derived exponential form Eq. (15). The cut off of inter-burst duration has the similar exponential form, but location is shifted proportionally to the h_x deviation from the mid point.

Observed power-law properties of this model of imitative behavior arise from the power-law properties of non-linear SDEs Eq. (10) extensively studied elsewhere []. Note that these properties are in close relation with rapidly developing ideas of non-extensive statistical mechanics and generalized concept of entropy [].

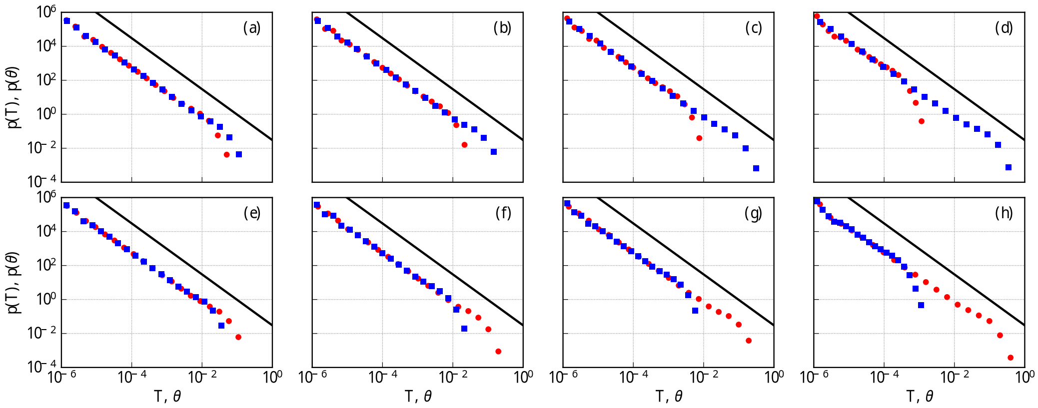

Figure 5. PDFs of burst (red circles) and inter-burst (blue squares) durations of time series, obtained by solving SDE (5) with μ given by Eq. (9), for various thresholds: $h_x = 0.6$ (a), 0.7 (b), 0.8 (c), 0.9 (d), 0.4 (e), 0.3 (f), 0.2 (g) and 0.1 (h). Parameter set used in numerical simulation: $\alpha = 2$, $\varepsilon_1 = \varepsilon_2 = 0$.

4. Conclusions

It is widely excepted that fluctuations of volatility and trading activity in the financial markets exhibit slowly decaying auto-correlation and $1/f$ noise [**? ? ? ?**]. The discussion whether this slow decay corresponds to long-range memory is still ongoing. The statistical analysis in general is not able to provide a definite answer concerning the presence or absence of long-range memory in finance [**? ? ?**]. From our point of view, the heterogeneous agent dynamics has to be employed seeking to explain statistical properties of financial time series [12**?** ,13]. Cirtainly, such complex system as finance [13] is not the best starting point for conceptual consideration of the long-range memory problem in other social systems. Thus in this contribution we consider much more abstract definition of agent system with imitative behavior leading to the continuing non-equilibrium stochastic fluctuations. Derived SDEs driven by Wienner noise describe Markov processes and can't be treated as suitable to model long-range memory with correlated stochastic increments. Such modeling by stochastic agent systems becomes as an alternative to the stochastic processes driven by fBm. Thus the choice between these two possibilities is the fundamental question for understanding of the observed long-range memory property in many other complex systems.

The model we investigate here is the generalized version of Kirman's herding model with pairwise global interaction of agents and is directly related to the voter model as well. The introduced additional feedback of macroscopic state on the time scale of interactions lets us to increase the non-linear multiplicativity η of SDE, defining properties of PSD for the ratio $y = \frac{x}{1-x}$. The retained symmetry of generalized equations makes this choice as preferable among other possible alternatives. From our point of view such modeling first of all is applicable to the financial systems, but is general enough and analytically tractable for other systems with heterogeneous agents.

Here we prove analytically and numerically that PDF of burst and inter-burst duration of stochastic variable y is a power-law 3/2 with exponential cut off for high values of durations. We consider this property as very valuable being very different from fBm process, where PDF of first passage time is dependent on H, $p(T) \sim T^{H-2}$ [19,20]. Thus the detailed empirical analyzes of burst and inter-burst durations may serve as a criteria to distinguish two different origins of $1/f$ noise and long-range memory property. Empirical evidence of the power-law exponent 3/ should be considered as a case of spurious memory, when deviations from this exponent should witness presence of real long-rage correlations. From our point of view the financial markets have to be considered as the social system with imitative behavior and spurious memory arising from the non-linear agent stochastic dynamics [12,13].

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Abbreviations

The following abbreviations are used in this manuscript:

PDF	Probability density function
SDE	Stochastic differential equation
PSD	Power spectral density

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