How Algebra, But Not Geometry, Is Based on Happy Coincidences

Abstract

This paper demonstrates why algebra cannot be used to instruct students in logic and rhetoric because of its heavy reliance on the algebraist having a lucky insight in which he spots some happy coincidence and, without providing any explanation for it, calls it a theorem and congratulates himself on his "analysis."

Suppose that you are asked to inscribe a regular octagon in a square. Unsure of how to begin, you draw four segments cutting out the corners at $45°$ angles and ask yourself, what must be true of these cuts for the figure to be a regular octagon? The angled segments must be equal to the uncut sections of the sides, between the cuts. And what are these lengths? If this is a unit square and what is cut from each side is of length x, then the uncut section is of length $1-2x$. The hypotenuse of the cut-out right isosceles triangle is $\sqrt{2}x$. Let's set them equal!

$\sqrt{2}x = 1 - 2x$	What must be true for the octagon to be regular
$2x^2 - 4x + 1 = 0$	Square both sides and collect like terms
$x = 1 - \frac{\sqrt{2}}{2} \approx 0.2929$	Quadratic formula; the other solution is too long

We could stop here, draw a 10 cm square and then use calipers to lay off 2.93 cm, but it would be better if we could relate this to the square somehow. x and $1-2x$ are collinear, so let's add them together: $\frac{\sqrt{2}}{2}$. Now we're cooking with gas! By a lucky insight, we recognize this as a number we have seen before. *But where???* Just by randomly scanning our eyes over the figure, we spot it: $\frac{\sqrt{2}}{2}$ is half of the square's diagonal. Even with no explanation for why these segments are equal, we then claim to have proven the inscribed octagon theorem. *What a lucky insight!!!*

Algebraists boast of this "elegant" construction as an example of their prowess in "analysis," but they nowhere establish cause and effect between the two lengths. They just happened to notice that one segment in their figure is labeled $\frac{\sqrt{2}}{2}$ and another segment has the same $\frac{\sqrt{2}}{2}$ label. My, what a happy and unexpected coincidence! This is considered a proof in algebra because $\frac{\sqrt{2}}{2}$ is the same length no matter where it appears. But, in geometry, we demand an explanation of cause and effect. The student cannot measure a length with his compass, randomly drop his compass around the figure until he finds another segment of the same length, and then shout *"Eureka!"* There are elegant proofs in algebra (e.g. $\sqrt{2}$ being irrational), but this is not one of them. *Common Core* shills like this proof because they have never studied geometry and they need an excuse to replace geometry with algebra. But the role that such happy coincidences play in algebra excludes algebra from instructing students in logic and rhetoric, which is our purpose.

Now let's do it right! A real mathematician does not accept two segments being the same length without an explanation of *why* they are. Rigor demands that we demonstrate cause and effect.

The following proof cites some basic geometry theorems:

1. Side–Angle–Side (SAS) Congruence
2. Isosceles Triangle Theorem Converse
3. Angle–Angle–Side (AAS) Congruence
4. Angle Sum Theorem
5. Isosceles Angle Theorem
6. Squares, Rectangles and Rhombi Theorem #1
   The diagonals of a square bisect each other and the vertex angles.

These are generally assumed of first-year geometry students; for reference, see *Geometry–Do*.[1]

Inscribed Octagon Theorem
Given a square with circles around each vertex of radii equal to half the diagonal, the circles cut the square at the vertices of a regular octagon.

Proof
Given $\overline{EFGH}$ square with center O, lay off $\overline{EO}$ on $\overrightarrow{EF}$ and $\overrightarrow{FE}$ to J and K, respectively. $\overline{EO} = \overline{FO}$ and $\angle OEJ = \angle OFK = \frac{\rho}{2}$ by the squares, rectangles and rhombi theorem #1; ρ is a right angle. By SAS, $\overline{OEJ} \cong \overline{OFK}$. By the isosceles angle theorem, their base angles are $\angle OKJ = \angle OJK = \frac{3}{4}\rho$. By the isosceles triangle theorem converse, $\overline{KJO}$ is isosceles; by the angle sum theorem, its apex angle is $\angle JOK = \frac{\rho}{2}$. By the isosceles angle theorem, the supplements of the base angles of $\overline{OEJ}$ and $\overline{OFK}$ are $\angle EKO = \angle FJO = \frac{5}{4}\rho$. By AAS, $\overline{EKO} \cong \overline{FJO}$ and, by the angle sum theorem, their apex angles are $\angle KOE = \angle JOF = \frac{1}{4}\rho$.

Lay off $\overline{EO}$ on $\overrightarrow{GF}$ to L. By an analogous construction, $\angle LOF = \frac{1}{4}\rho$ and $\overline{LO} = \overline{JO}$. By SAS, $\overline{JOK} \cong \overline{LOJ}$. Analogously, there are eight congruent triangles; thus, an octagon. ■

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[1] www.researchgate.net/profile/Victor_Aguilar4/publication/291333791_Volume_One_Geometry_without_Multiplication

This proof teaches logic and rhetoric far better than just saying, "Golly! Gee-whiz! This segment is the same length as that segment over there! Maybe there is like – *um* – some sort of connection between them. Let's call it a theorem and congratulate ourselves on our analysis!" Such reliance on happy coincidences is *not* elegant, whatever the *Common Core* shills might say.

To meet the severe teacher shortage in America, Traci Taylor[2] is recruiting at Starbucks!

> *At the grocery store, at Target, at Starbucks, anywhere I go, if I meet someone who seems smart and engaging, I give them my card and say, "Be a teacher!" It doesn't matter what your circumstances are.*

Why is there such a severe shortage of mathematics teachers? Heather Voke[3] explains:

> *Almost one third of all high school math teachers have neither a major nor a minor in math or a related field… One-fourth of all beginning teachers leave the classroom within the first four years… Even more alarming than the turnover rates themselves are data suggesting that the most intelligent and effective teachers leave the profession at the highest rates… new teachers who scored in the top quartile on their college entrance exams are nearly twice as likely to leave teaching than those with lower scores.*

Why are the good teachers leaving in droves? Because they are appalled by the incredibly low standards of *Common Core* mathematics. It is my contention that these low standards are a direct result of *Forum Geometricorum* promoting the idea that geometry is just a review of basic algebra that happens to be illustrated with triangles. Paul Yiu[4] writes:

> *Like handling difficult problems in synthetic geometry with analytic geometry, one analyzes construction problems by the use of algebra… For all the strength and power of such algebraic analysis of geometric problems, it is often impractical to carry out detailed constructions with paper and pencil, so much so that in many cases one is forced to settle for mere constructability… Geometer's Sketchpad™.*

I wrote a 300-page geometry textbook and I *never* settled for mere constructability. I have higher standards than that. Paper and pencil constructions are *easy* if you know what you are doing! Low standards are not *forced* on me by *McGraw-Hill* trying to sell *Geometer's Sketchpad™*. I am above plugging a commercial product in the abstract of a paper about elegance in mathematics.

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[2] www.edsource.org/2017/fresno-tackles-its-shortage-of-math-and-science-teachers/581342

[3] www.ascd.org/publications/books/104138/chapters/Responding-to-the-Teacher-Shortage.aspx

[4] www.forumgeom.fau.edu/FG2005volume5/FG200512index.html

But, if Paul Yiu likes *McGraw-Hill* so much, then let's have a look at their geometry textbook.

In America, geometry is a sophomore class that comes between Algebra I and II. *Common Core* turns triangle congruence into a review of Algebra I; the congruent triangles are just an excuse for attaching linear equations to two lengths, two angles or – *God forbid!* – a length and an angle. This is wrong on so many levels! (1) It ignores real geometry; (2) It is bad Algebra I to add lengths and angles; and, (3) Cramer's Rule is in Algebra II, so it is taught after it is needed.

Glencoe Geometry (p. 256) declares two triangles congruent, one with all its sides and angles labeled: $a = 38.4$ mm, $b = 54$ mm, $c = 32.1$ mm and $\alpha = 45°$, $\beta = 99°$, $\gamma = 36°$. The other triangle has the side corresponding to a labeled $(x + 2y)$ mm and the angle corresponding to β labeled $(8y - 5)°$. *Glencoe* solves $x + 2y = 38.4$ mm and $8y - 5 = 99°$ simultaneously to get $x = 12.4$ and $y = 13$. This is stupid – *Look at the units!* – but nobody cares because Paul Yiu has convinced people that geometry is just a boring review of basic algebra, illustrated with triangles.

Indeed, it was page 256 of *Glencoe Geometry* that ended my career as a geometry teacher. I told my students that there was a mistake in their textbook and was pointedly informed by the administration that subs – *Ahem!* – paraprofessionals do not have the authority to question their "fine" *Common Core* textbooks. So now I am a textbook author; we will see how well that goes.

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