Towards an Exact Solution of the American Style Options III.

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Abstract:

We have recently reported on the issue of uncertainty of exercise inherent to the American style option. This uncertainty results in an inequality between the portfolio and its constituent option and assets (etc.). We have approached this from the point of view of a random boundary in time problem by Green's functions methods in I, and more recently as a generalization of the linear programming approach where equalization of the otherwise inequality of the portfolio relation problem is made by introducing a slack function which we identify with the uncertainty of exercise.

In this letter we continue this second approach however from a combined portfolio Black-Scholes approach coupled with an information theoretic and equivalently (maximum) entropy of thermodynamics Hamiltonian superposition view, and the Gibbs-Bogoliubov inequality approach.

Introduction:

The Black-Scholes equation can be derived from one approach by positing a portfolio relation as

$$dx = a(x,t)dt + b(x,t)dW(t)$$
$$\Pi(t) = f_o(x,t) - \Delta x$$
$$d\Pi(t) = r\Pi dt$$
$$d\Pi = df_o - \Delta dx$$
$$\frac{\partial f_o}{\partial t} = r(f_o - x\frac{\partial f_o}{\partial x}) - \frac{b^2(x,t)}{2}\frac{\partial^2 f_o}{\partial x^2} \quad (1)$$

where the backwards Fokker-Planck PDE is derived by variation and a choice for the delta and the r rate of portfolio wealth increase. Here f_o denotes the Black-Scholes fixed exercise time horizon European style option. Note the diffusion coefficient b is here arbitrary and may correspond to the lognormal underlying stochastic process or may be more complex such as the Tsallis-Zanette-Borland q-parameterized-PDF containing form recently applied with great accuracy to real markets, and whereas the drift coefficient a(x,t)=ux of a traditional B-S log-normal type if b(x,t)~x in the financial market underlying, is removed as usual in favor of the desired r dependent drift.

This general B-S Black-Scholes process and the derived model equation can be generalized to other statistics as mentioned, such as the nonextensive statistics processes readily and this has been reported elsewhere.

We keep the simple B-S extensive forms in this following derivation and for simplicity of illustration of the methods we utilize, and consider generalizing to more complex, such as the more accurate nonextensive statistics, the 'American style' model of derivative options in future work.

The American style option arbitrary exercise time introduces additional uncertainty to an already diffusive or stochastic process such as Eq.(1), and which can be accounted for in one approach by noting that the portfolio relation and therefore derived PDE structure is now an inequality:

$$\begin{aligned}
&\Pi(t) \geq f_{o/\sim}(x,t) - \Delta_{o/\sim} x \\
&d\Pi(t) = r\Pi dt \\
&d\Pi \geq df_{o/\sim} - \Delta_{o/\sim} dx \\
&\frac{\partial f_{o/\sim}}{\partial t} \leq r(f_{o/\sim} - x\frac{\partial f_{o/\sim}}{\partial x}) - \frac{b^2(x,t)}{2}\frac{\partial^2 f_{o/\sim}}{\partial x^2}
\end{aligned} \quad (2)$$

where the option here in the portfolio inequality is referenced to f_o/~ the European style derived type of option f_o or some trial or toy model f_~ . These (or any instrument that..) not fully accounting for the exercise uncertainty of an American style option. We assume the 'fully-accounting-for' early or arbitrary exercise American option f and its modified 'full' hedge will derive an equality portfolio relation, whose underlying i) asset or security dx=adt+bdW equation maybe known, however ii) its early exercise arbitrariness and therefore uncertainty, if written as du=cdt+vdQ, is not known.
This is usually not explicitly stated in the traditional approach however that is what the inequality means. The American style portfolio due to the uncertainty associated with arbitrariness of exercise is unequal between risk free or desired rate of value evolution and its constituents as traditionally derived from purely market uncertainties, obtaining an inequality. Certain exercise time and therefore non-random in time

boundaries of the European style portfolio results in an equality between the portfolio and its composition of options and assets and/or securities...bonds, etc...

The American style option value approaches the European style value as time approaches the exercise time (fixed, European), equivalently the inequality portfolio approaches this equality (portfolio relation) in value only as the time approaches the European exercise time, the two equating at that time. Again this in the traditional approach, which is only (merely) dependent on the market uncertainty of the underlying.

Therefore we have concluded and pursued the supposition that defining the referenced option as the European style option or some toy model, from which additional components approach the full or American style option valuation, preserve this limiting approach, and in fact which may even before the European exercise time equality reach the desired equality if by inclusion of components compensating for the sources of uncertainty, is a reasonable assumption to make and to apply.

As an aside and to summarize some recent results, we state the problem in terms of stochastics and PDEs in Appendices A-C.

The considerations discussed thus far, we assert, imply that there is some exact American style option and portfolio relation that is an equality!

$$\begin{aligned}\Pi(t) &= f(x,t) - \Delta_f x \\ &= f_o + \delta f_o - \Delta_{f|_{f_o+\delta f_o}} x \\ &= f_\sim + \Delta_{f|_{f_\sim}} x\end{aligned} \quad (3)$$

and where the precise description of the American style derivative (option) f and the effective hedge's delta itself depending on f is yet to be determined, and is not immediately obtained by the traditional and by considering market-only uncertainties and the market underlying only dependent approach. Said another way, any additional exercise uncertainty components beyond the equality market uncertainty components that may allow or which may result in an equality of the portfolio relation are 'absorbed' or are contained in the constituents f and delta these cast however in an effectively traditional portfolio form.

However this is the very issue, the determination of the existence of such an equal relation Eq.3 and then the precise determination of its constituents. Also, the hypothetical equalities in Eq.(3) are merely suggestive at this point and imply the search for variations or superposition away from European style options, and away from (traditional) delta hedges, and away from (known, equality portfolio relation) toy or trial ~-twiddle models, these to be discussed further.

In order to pursue the precise definition of f and delta and the hypothetical equality portfolios for American style derivatives options, we wish to define the 'mapping' of the inequality and equality, below portfolio relations, to the thermodynamics and Gibbs-Bogoliubov inequality arguments, and furthermore then mappings to the maximum entropy equivalently information theory approaches we will be utilizing in order to investigate the additional components.

Importantly, first note the following. The options instrument(s) is described by f_o or f_~ or fully f etc. and where we state that these are a type of probability based distributions, observables' density functions in detail, here proportioned as evolution distributions of monetary value or price distributions.... therefore these are in our view statistical weights multiplied by a monetary value. Alternatively, these observable's distributions are proportioned such that they evolve as distribution (statistical) functions evolving as by the variables of the observables...any observable then is obtained by averaging its value with these distributions weighting the integration or summation as <x> etc.

However the option distribution or pricing function is not in the standard form we would like to utilize. First the backwards Fokker-Planck of Black-Scholes should be considered from the forward p.o.v. point of view. This is most easily accomplished by considering that the two point function form of the option distribution f(x,t|x',t') solves both the backwards and forwards Fokker-Planck equations and when the relation f=e$^{()}$g(x,t|x',t') is utilized and a shift in variables emphasis to forward is made (dropping prime superscripts, using these interchangeably by context throughout for simplicity), we have

$$< S_o >_o = -\int g_o \ln g_o(x,t)dx = -\int g_o[\ln(\Pi(t)+\Delta x)+\ln e^{r()}]dx$$

$$< F_o >_o = -\frac{1}{\beta_o}\ln\int g_o dx = -\frac{1}{\beta_o}\ln\int(e^{r()}(\Pi(t)+\Delta x))dx \quad (3)$$

and where we can work with f or g (generally) notationally yet in the standard forward Fokker-Planck form for consistency and connection to Hamiltonians and thermodynamics, our guiding physical and mathematical theory. We will replace exponential prefactors & forward or backward notation upon derivation. At times we retain the f notation yet we mean the standard forward g evolution consistent with Fokker-Planck to Diffusion-Schroedinger (such as the Matsubara-Schroedinger picture) Hamiltonians. Also, we focus on the relevant scale of the physical observable dynamics for the Hamiltonian formulation, room or human trader markets' scale diffusion.

Upon putting the PDE for g in the standard forward form, we immediately note the following thermodynamics (or if negentropy then information theoretic related) relations:

$$F_o = \langle F_o \rangle_o = -\frac{1}{\beta_o} \ln Z_o$$

$$Z_o = \int g_o(x,t)dx$$

$$S_o = \langle S_o \rangle_o = -\int g_o \ln g_o dx \qquad (3)$$

these one-point relations, and we take care to denote the level of averaging per which ensemble or system, for example <>_o denoting averaging over the European style (say..) statistics g_o, and where the observable to be averaged itself may belong to the European model or a trial or toy model or the American style 'full' model, these which we will annotate differently.

The two-point (x;y) joint and/or conditional relations (x|y), and these conditionals when temporal (example (x,t|x',t') x>x' and t>t' or t=t'), also known as transition probabilities or distributions, and the 'conditional expectation values' such as for example the definitions of the 1st and second etc.
Conditional moments of diffusion processes, which also are obtained as from two-point functions, and which are defined similarly (conditional moments discussed elsewhere):

$$Z_o(t \mid t') = \int g_o(x,t \mid x',t')dxdx'$$

$$Z_o(x',t' \mid t) = \int g_o(x,t \mid x',t')dx$$

$$S_o = \langle S_o \rangle = -\int g_o(x,t \mid x',t')\ln g_o()dxdx'$$

$$S_o(x',t' \mid t) = \langle S_o \rangle = -\int g_o(x,t \mid x',t')\ln g_o()dx \quad (4)$$

and where we will refer to these forms as necessary and utilize notation as needed to distinguish and per context.

Consider as examples of these 2-point considerations that the drift and diffusion coefficients as we mentioned are defined as the conditional first and second moments respectively, and where we have already had cause to utilize the fact that the symmetrical two-point function solves the forward and backward Fokker-Planck PDE. Additionally, state functions such as entropy, internal energy etc. at the one-point level can be extended to two or more points functions or conversely, if integration of one of the two or more points is made, remaindering the one-point functions.

In the above equation Eq.(3) F is the free energy, Z the partition function (inverse normalization N=1/Z), S the Gibbs-Boltzmann type of extensive entropy here or its negative (with proper constants <S>=-<I> Shannon information), and we retain the traditional mixed confusion of notation between state functions and averages of microscopic quantities e.x. S=<S> which obtain these macroscopic state functions if only to remind the familiarity of the reader with these notational issues. Also, we have set k_B to unity above and in the following, and where the Lagrange multiplier beta_o is as usual the inverse temperature in thermodynamics, and the inverse variance and beyond that inverse conditional 2nd moment and further interpretations not considered here, in statistical x,t descriptions as also from information theory. And this also applicable in the corresponding thermodynamics and the maximum entropy method we will utilize later and where the Lagrange multiplier(s) enters as a proportionality factor(s) into the Legendre transformations and extremizations we will apply.

Continuing to define the mapping to thermodynamics and to information theoretic approaches utilizing the European style instrument as an example but a method which we will apply to the toy or twiddle trial instrument and the inequality American style portfolio, we transform the portfolio relation(s) to these as

$$< S_o >_o = -\int g_o \ln g_o(x,t)dx = -\int g_o[\ln(\Pi(t)+\Delta x)+\ln e^{r0}]dx$$

$$< F_o >_o = -\frac{1}{\beta_o}\ln\int g_o dx = -\frac{1}{\beta_o}\ln\int(e^{r0}(\Pi(t)+\Delta x))dx \quad . \qquad (5)$$

and in the following we will drop the r,t dependent prefactor exponential as it merely adds a time dependent term and we can reinstate it as we can any function or constant of proportionality or of addition in later comparisons and re mappings to portfolios.

And we recall that we are utilizing in Eq.(5) etc. the European style option and similarly therefore also the equality obtained for that type of portfolio relation, and that the hypothesized toy or trial option also obtains an equality portfolio models. The European style option f_o (g_o) with corresponding Hamiltonians H_o, this as the reference state or 'known' model, towards or for the description and modeling of the full inequality American portfolio Black-Scholes -like portfolio.

As an example of the approach we will utilize, and we have shown how the standard form Fokker-Planck' PDF f_o etc. of the Black-Scholes' model (implied from ..) maps or is related to thermodynamics and due to the equivalence of entropy and its negative (the 'negentropy' with proper proportionality constants) also the C.Shannon information measure of information theory, though we will as other authors use these interchangeably... the signs & constants assumed into the Lagrange multipliers. Maximum entropy being traditionally a 'physical theory' adding experimentally observable factors, while information theory also considering known and unknowns and considering issues of an incomplete knowledge of a system, these perhaps philosophical issues now coming to bear upon and we consider becoming very relevant to our 'suppositions' and our 'approaches' in our current problem.

The thermodynamics relations of interest to us here are the usual free energy and entropy relations:

$$< F_o >_o = < U_o >_o - T_o < S_o >_o$$
$$U_o = < U_o >_o = < H_o >_o \qquad (6)$$

where for this European style thermodynamics (!) we explicitly wrote the statistics utilized in the averaging, and where 'naught' the subscript of say F_o or <..>_o refers to the observable of averaging of the (known system) problem of the European style Black-Scholes. As an aside, we state that we can delve into deeper microscopic levels, continuous and discrete and even the so-called 'quantum portfolios' problems of finance and quantum computing via ensembles of Qbits, for the definition of H and its (say, ensemble) averaging.

For our illustrative approach we continue with the diffusion approximation continuous Fokker-Plank & analogous Schroedinger-thermal (a.k.a. Matsubara) diffusion approximation PDEs level of description.

And also, we should mention that we can utilize apriori nonlinear composition of entropy and which derive the required statistics. The q-parameterized nonextensive C.Tsallis statistics for example have with exquisite accuracy described the statistical dynamics of real financial markets and the stochastic evolution of these markets and have led to and with similarly great accuracy, derivatives pricing formulas recently. And these we will have the opportunity to utilize in later work to parallel our early generalized nonextensive Derivatives pricing or generalized q-Black-Scholes formula(s).
That is we will, if successful in deriving a proper American style Derivatives pricing formulation, immediately set about generalizing its minutiae to the nonextensive statistics of course.

The internal energy <U> in Eq.(6) is equated to the Hamiltonians <H> for the Black-Scholes process. At the level of the diffusion approximation which obtains 'continuous' PDEs such as the Fokker-Planck, this is the as usual the resulting, transformed to Schroedinger-like PDEs' kinetic energy plus the potential energy, with the potential energy especially obtainable from the forward PDE's steady-state as

$$(-rx)g_o = \frac{1}{2}\frac{\partial b^2(x,t)g_o}{\partial x}$$

$$h_o = b^2 g_o$$

$$2(-rx/b^2) = \frac{\partial \ln h_o}{\partial x}$$

$$h_o = e^{-2r\int^{x}\frac{x"}{b^2(x",t_o)}dx"} \qquad (7)$$

where the general potential with b not time dependent as in the log-normal Black-Scholes model say.., is drift and diffusion coefficient dependent in the integral. As an Alternative p.o.v., one can transform the Fokker-Planck equation to a thermal (Matsubara type) Schroedinger equation and simply read off the resulting potential. The point being that the Hamiltonian H_o is well defined and known for the European B-S in the above relations.

To elaborate, the Hermitian Hamiltonian operator of a Schroedinger equation can be obtained by a series of transformations; to summarize these, the Hamiltonian operator of the PDE is (to within constants and prefactors)

$$\frac{\partial \phi}{\partial t} = \hat{H}\phi$$

$$\hat{H} = (\frac{a'}{2} - \frac{a^2}{4b^2}) + b^2\frac{\partial^2}{\partial x^2}$$

$$\hat{L}\rho = \sqrt{\rho_{t_o}}\hat{H}\phi$$

$$\rho_{t_o} = Ce^{-\int\frac{a}{b^2}dx"} \qquad (7.b.)$$

With <H>=<KE> + <PE> the superposition of kinetic and potential energy and L the F-P operator.

We can make the H Hamiltonian simpler, this simplicity to aid us in the derivations to follow.

The log-normal Black Scholes PDE's g_o equation PDE has as stated an implied risk free rate drift log-normal stochastic SDE process

$$dx = -r dt + \sigma dW$$
$$x = \ln S$$
$$y = x - rt$$
$$dy = +\sigma dW$$
$$\partial_t g_o(y,t \mid y',t') = \frac{\sigma^2}{2}\partial_y^2 g$$
$$\partial_t g_o = \hat{L}_o(y) g_o = \hat{H}_o(y) g_o \qquad (7.c.)$$

this allows us to transform to a diffusion only frame in y,t & therefore the Fokker-Plank operator L corresponds to the Hamiltonian operator.

Now we make an important supposition. This supposition will define the remaining derivations. We suppose that the full and unknown Hamiltonian Eq.(7-7.b.) H is referenced to the European style and the known Black-Scholes Hamiltonian H_o. First the American style Hamiltonian is written as

$$H = H_o + \delta H$$
$$\delta H = H_1 \qquad (8)$$

and then we define a trial Hamiltonian that we can manipulate by expansion or superposition or such, to approach possibly in some limit and however on the average, the full Hamiltonian H, this written such that its relation with the full Hamiltonian is defined as

$$\tilde{H} = H_o + < \delta H >_o$$
$$(\delta H \to H_1)$$

$$<\tilde{H}>_{\sim}=<H_o>_{\sim}+<<\delta H>_o>_{\sim}$$
$$<H>_{\sim}=<H_o>_{\sim}+<\delta H>_{\sim}$$
$$<\tilde{H}>_{\sim}=<H>_{\sim}+<<\delta H>_o>_{\sim}-<\delta H>_{\sim}$$
$$<<\delta H>_o>_{\sim}=<<\delta H>_{\sim}>_o=<\delta H>_{\sim}$$
$$<\tilde{H}>_{\sim}=<H>_{\sim} \quad . \qquad (9)$$

which depends on the interchangeability of the averaging and then further on the equivalence of the averaged observables.

To continue, the Free energy associated with these Hamiltonians obtains the Gibbs-Bogoliubov inequality:

$$<\tilde{F}>_{\sim}\geq<F>_{\sim}$$
$$\tilde{F}\geq F$$
$$<F>_{\sim}\leq \ <H>_{\sim}-T<\tilde{S}>_{\sim} \qquad (10)$$

this a statement that the free energy of the full (unknown) system is less than or equal to the thermodynamically averaged full <U>_~ components under the known system's ensemble averaging & by the relation with (known, ~) entropy.

We obtain Eq.(10) when we examine the model and its averages <..>_~ and the full Hamiltonian and its averages and we combine (9) and (10) we obtain the inequality as averaged over the model H_~ and this referenced by the European style H_o

$$<\tilde{F}>_{\sim}=<\tilde{H}>_{\sim}-<\tilde{S}>_{\sim}$$
$$<\tilde{F}>_{\sim}=<H>_{\sim}-<\tilde{S}>_{\sim}$$
$$<F>_{\sim}\leq \ <H>_{\sim}-<\tilde{S}>_{\sim} \quad . \qquad (11)$$

Thus far the above inequality is merely the Gibbs and Bogoliubov inequality. We wish to utilize this to connect the various perspectives of analytics of the European with the American style options.

We remind the reader that in our previous, recent letters (Towards an Exact Description of the American Style Options I-II), we had in II 'added' uncertainty contributions to the portfolio inequality these additional sources of uncertainty due to the random in time boundary experienced by the evolution of the option price which we examined via Green's functions in I, deriving even there an admittedly complicated expression for the American style option while the II saw the uncertainty contribution as merely an additional diffusion component (we chose this component as 'diffusion-only', as drift+diffusion can be apportioned to one or the other or between both simultaneously), rendering the resulting portfolio as an equality portfolio & F-P via the additional (averaged) uncertainties which we labeled in II as generalized 'slack functions' and/or additional stochastic/diffusion sources, these additional components transforming portfolio inequalities to equalities (supposedly! thus the statement about philosophies, mathematics and physics and incompleteness of information, and in fact this letter seeking to establish the validity of these suppositions from yet another perspective that of thermodynamics etc.) along the lines of reasoning of optimization and linear programming problems approaches and of stochastics and PDEs.

We claimed in I-II that the additional uncertainty was composed generally of a drift coefficient and diffusion coefficient but we chose simply the diffusion; this yet supposedly fully descriptively as due to the freedom of drift+diffusion proportioning of the underlying stochastic trajectories, and this 'fully' in the sense of obtaining portfolio equalities. From E.T. Jaynes' perspective, perhaps a better word would be 'sufficiently'. The diffusion-only form of the exercise uncertainty which generally is drift+diffusion as for any source of open system uncertainty or stochastics, served to illustrate.

This additional term which is presumably supported by otherwise generally valid arguments from superposition, from Legendre transformations' additions of observables (E.T.Jaynes & C.Shannon) weighted by Lagrange multipliers as in maximum entropy and equivalently information theory's addition of observables or known or surmised effects. And given the previous (I) analysis of random in time boundary conditions via Green's functions and conclusions gleaned from that approach as applied to the problem in II.

In this approach however, of thermodynamics and Hamiltonians and Gibbs-Bogoliubov inequalities, we are seeking to analyze inequalities of portfolio compositions, that are due to the underlying uncertainty of exercise, in light of and by equating with additionally the uncertainty of 'trial' model(s) as compared to a full model, that is by inequalities of the Gibbs-Bogoliubov type which we state: that these two inequalities are equatable!

And we seek any insight to be gained from Gibbs-Bogoliubov inequality relations, these applicable to uncertainties in full descriptions of say full H and as compared to or approximated by trial or model Hamiltonians which then approach the full model by some scheme (say expansions and parameterizations in expansions or ad hoc or q-nonextensive or superposition) can shed light upon the second uncertainty, the non-market arbitrariness-of-exercise source's uncertainty structure and the resulting portfolio constituents of the American style options problem.

Therefore let us describe again the problem from portfolio and statistics/stochastics and from Hamiltonians and thermodynamics/informatics and G-B inequalities.

The inequality portfolio is written in terms of free energy as

$$Z = \int f dx$$

$$Z_{\sim/o} = \int f_{\sim/o} dx$$

$$< F >_{\sim} = -\frac{1}{\beta_{\sim}} \ln < f >_{\sim} \leq -\frac{1}{\beta_{\sim}} \ln \int (\Pi_{\sim} + \Delta_{\sim} x) dx$$

$$< F >_{o} = -\frac{1}{\beta_{o}} \ln < f >_{< f > = < f >_o} \leq -\frac{1}{\beta_{o}} \ln \int (\Pi_{o} + \Delta_{o} x) dx \qquad (12)$$

and where the corresponding models are related as before, that is by the averaged macroscopic level, and in detail by the model's choice of option f, f_o or f_~ and full , twiddle (toy or trial) and naught or European Hamiltonians and furthermore by the Lagrange multiplier beta corresponding to these...we will discuss this further.

The definition of Free energy F~lnZ being however proportional to the logarithm of the partition functions Z, Z_o or Z_~ in the portfolios picture and its thermodynamics picture as well as the x,t PDE/SDE statistics/stochastics picture, and again furthermore depending in the inequality on the form of Delta hedge this also delta, delta_o or delta_~, these having the usual form of the partial differential of the derivative w.r.t. the underlying of the appropriate function.

The usual way the Gibbs and Bogoliubov inequality is applied to real problems is to determine the maximum of certain parameters dependent on the form of the problem such that the model and full Free energy become equal.

This is made possible by the relations discussed above, and the guarantee of an upper bound etc. equating the two free energies and from the equality of averages of the Hamiltonians.

We will in this letter and initially merely utilize and apply the method, to examine the additional variation of extra Hamiltonian components in later work. We will however examine what it means to add by superposition components, as this in Hamiltonian language (we claim) is the analog of adding additional sources of uncertainty due to arbitrariness of exercise as we did in II.

And we then wish to compare this with the imports of letters II as suggested by letter I, that the compensatory terms in the inequality which then result in an equality portfolio and dynamics are additional sources of uncertainty due to arbitrariness of exercise and are described by drift and diffusion components.

Obviously with the mapping between thermodynamics and finance as in the above discussion, other and possibly more accurate forms of descriptions of the uncertainty due to arbitrary exercise time will be possible and potentially more preferable. The arbitrary exercise time seems well suited for example for modeling by a history following process, or nonextensive statistics & nonextensive thermodynamics.

In order to continue towards the desired results, we rewrite the full Hamiltonian as before in terms of the model Free energy and the portfolio constitutive relation,

$$< \tilde{F} >_{\_} = < H >_{\_} - < \tilde{S} >_{\_} \tag{13}$$

From equations Eq.(11) and Eq.(12) we can now equate the thermodynamic expression for free energy and the portfolio:

$$
\begin{aligned}
&< F >_{\sim} \leq \; < H >_{\sim} - < \tilde{S} >_{\sim} \\
&< F >_{\sim} = -\frac{1}{\tilde{\beta}} \ln < f >_{\sim} \leq -\frac{1}{\tilde{\beta}} \ln \int (\Pi + \tilde{\Delta} x) dx \\
&< H >_{\sim} - < \tilde{S} >_{\sim} = -\frac{1}{\tilde{\beta}} \ln \int (\Pi + \tilde{\Delta} x) dx \\
&< H >_{o} - < S_{o} >_{o} = -\frac{1}{\beta_{o}} \ln \int (\Pi + \Delta_{o} x) dx
\end{aligned}
\tag{14}
$$

Note that in Eq.(14) therefore we have equated the full American style Hamiltonian operator in terms of the known model & therefore European style model to the known model (twiddle) portfolio relation. Thus we assume that by modifying the additional Hamiltonian component we can make such an equality.

Yet a strictly Gibbs-Bogloiubov equation is obtained additionally with this assumption. Consider that due to the equality of averages of Hamiltonians, and forcing the equality of free energies we have at the upper bound

$$< F >==< \tilde{F} >_{-}$$

$$-\beta^{-1} \ln Z == -\tilde{\beta}^{-1} \ln \tilde{Z}$$

$$Z = e^{\frac{\beta}{\tilde{\beta}}} \tilde{Z}$$

$$f = e^{\frac{\beta}{\tilde{\beta}}} \tilde{f} \qquad (14.b)$$

which when the portfolio pictures are utilized obtain

$$\int (\Pi + \Delta x - e^{\frac{\beta}{\tilde{\beta}}} (\tilde{\Pi} + \tilde{\Delta} x)) dx = 0$$

$$\int \frac{\partial f}{\partial x} x dx = [xf]_{x_o}^{x_f} - \int f dx$$

$$\int (\Pi + bc - f - e^{\frac{\beta}{\tilde{\beta}}} (\tilde{\Pi} + bc_{\sim} - \tilde{f})) dx = 0$$

$$f = \Pi + bc - e^{\frac{\beta}{\tilde{\beta}}} (\tilde{\Pi} + bc_{\sim} - \tilde{f})$$

$$f = e^{\frac{\beta}{\tilde{\beta}}} \tilde{f} + \Pi (1 - e^{\frac{\beta}{\tilde{\beta}}}) + bc - e^{\frac{\beta}{\tilde{\beta}}} bc_{\sim}$$

$$\Pi (1 - e^{\frac{\beta}{\tilde{\beta}}}) + bc - e^{\frac{\beta}{\tilde{\beta}}} bc_{\sim} \rightarrow 0 \qquad (14.c)$$

where we utilized the definition of the basic continuously adjusted delta hedge which when we equate the Eqs.(14.b, 14.c) requires that the portfolio+boundary conditions must vanish to force the equality this whether we choose the same rate of profit of portfolios in both or retain different profitability (r known or desired rate) portfolios. This suggesting that generalizing the portfolio side in order to equate or force the equating of the inequality portfolio is a potential path to valuation however we will investigate this in later work.

From the Hamiltonian equality of averages by the twiddle ensemble, we have

$$< H >_{\sim} = < \tilde{H} >_{\sim}$$

$$d\int H\tilde{f}dx = d\int \tilde{H}\tilde{f}dx$$

$$\int Hd\tilde{f}dx = \int \tilde{H}d\tilde{f}dx$$

$$\int H\frac{\partial\tilde{f}}{\partial f}dfdx = e^{\frac{\tilde{\beta}}{\beta}}\int Hdfdx =$$

$$d < H >= e^{\frac{\beta}{\tilde{\beta}}}d < \tilde{H} >_{\sim} = e^{\frac{\beta}{\tilde{\beta}}}d < H >_{\sim}$$

(14.d)

The toy model merely a time dependent factor away from the European model, the two Lagrange multipliers of twiddle and European can be also determined and their relationship detailed. This is possible due to normalizations or their inverses the partition functions (we will discuss this further), and where the two partition functions of twiddle and European free energies differ by the time dependent only factors.

In the stead of rederiving from free energy and portfolio pictures in order to determine further the twiddle and European relations, let us utilize the identity

$$\beta_o Z_o^2 = c_o$$

$$\tilde{\beta}\tilde{Z}^2 = \tilde{c}$$

$$\beta Z^2 = c$$

$$let\ \tilde{c} = c_o h(t)$$

$$\tilde{Z} = \sqrt{\frac{\beta_o}{\tilde{\beta}}h(t)}Z_o$$

$$\tilde{f} = \sqrt{\frac{\beta_o}{\tilde{\beta}}h(t)}f_o \qquad (14.e)$$

and therefore we have an expression for the full derivative in terms of the twiddle derivative and even the European derivative by combining the series of equations of Eq.(14). These may serve as points of departure and generalization or as consistency checks when we apply the import of the letter that of adding (adhoc, Legendre transforms, superposition...) additional sources of uncertainty corresponding to early and arbitrary exercise and deriving Fokker-Planck PDEs from that.

Let us elaborate on Eq.(14.e). The c_o is literally a constant. This is clearly seen by considering the pure diffusion (in y,t) of Eq.(7.c) which results in the Lagrange multiplier and partition function as

$$\beta_o(t) = \frac{1}{2\sigma_o^2 t}$$

$$Z_o(t) = \sqrt{4\pi\sigma_o^2 t}$$

$$\beta_o Z_o^2 = c_o = 2\pi \qquad (14.f)$$

while the PDE of the twiddle portfolio acquires a time dependent averaged Hamiltonian component <H_1>_o , this precisely related to h(t) in the above equations and entering therefore into the relation between twiddle and naught identities and the derivatives which comprise these.

Specifically, the twiddle PDE from twiddle max entropy least biased PDF is discussed further, from one Lagrange, pseudo two Lagrange and even two Lagrange multiplier Legendre transform proportioning:

$$d < \tilde{S} >_{\sim} + d[\tilde{\beta} < \tilde{H} >_{\sim}] - d[\tilde{Z}] = 0$$

$${}_1\tilde{f} = e^{-\tilde{\beta}\tilde{H}} / {}_1\tilde{Z}$$

$$d < \tilde{S} >_{\sim} + d[\tilde{\beta}(< H_o >_{\sim} + \tilde{\alpha} << H_o >_o >_{\sim})] - d[\tilde{Z}] = 0$$

$${}_2\tilde{f} = e^{-\tilde{\beta}(H_o + \tilde{\alpha} < H_o >_o)} / {}_2\tilde{Z}$$

$$d < \tilde{S} >_{\sim} + d[\tilde{\beta} < H_o >_{\sim} + \tilde{\alpha} << H_o >_o >_{\sim}] - d[\tilde{Z}] = 0$$

$${}_3\tilde{f} = e^{-\tilde{\beta}H_o - \tilde{\alpha} < H_o >_o} / {}_3\tilde{Z} \qquad (14.g)$$

the point is clearly apparent, the normalizations and Lagrange multiplier identification are dependent upon the partitioning of the proportionalities that the Lagrange multipliers are.

The simplest to manipulate is (3) in Eq.(14.g)...assume that the H_o Hamiltonian's 'partitioned' proportionality multiplier is identified with the naught beta_~ -> beta_o multiplier and we can write ('un' is the yet unnormalized least biased PDF of (3))

$${}_3\tilde{f} = e^{-\tilde{\beta}H_o - \tilde{\alpha} < H_1 >_o} / {}_3\tilde{Z}$$

$${}_{un}\tilde{f} = e^{-\beta_o H_o - \tilde{\alpha} < H_1 >_o}$$

$$\tilde{Z} = \int {}_{un}\tilde{f} dx = e^{-\tilde{\alpha} < H_1 >_o} \int e^{-\beta_o H_o} dx =$$

$$\tilde{Z} = e^{-\tilde{\alpha} < H_1 >_o} Z_o$$

$$-\frac{1}{\tilde{Z}}\frac{\partial \tilde{Z}}{\partial \tilde{\alpha}(t)} = < H_1 >_o \qquad (14.h)$$

and where alpha_twiddle can often be determined easily (here) from the Lagrange multiplier identity, yet at times one must resort to the PDE and SDE levels or even the thermodynamics relations... the Eq.(14.h) illustrates the points above, regarding relationships between full, twiddle and naught partition functions, multipliers and derivative functions.

Also in Eqs.(14) we can rearrange and absorb any/all constants and pre-factors into the Lagrange multiplier(s) which we will re-introduce upon derivation of relevant solutions.

The Hamiltonian models maximum entropy with additional portfolio constraints, additional information:

The reader may choose to read past this section initially, as it is more of an investigation of constraints rather than the import, that of exercise uncertainty modeling.

To return to the Free energy description of thermodynamics as relevant to each level of the portfolio, we had at the twiddle model description level

$$< \tilde{S} >_{-} + \tilde{\beta}\{< H >_{-}\} + \tilde{\alpha}\{\ln \int (\Pi + \tilde{\Delta} x) dx\} = 0 \tag{15}$$

and whereas the expression immediately reminds of the maximum entropy method for derivation of the least biased statistics or distributions here, we vary with the additional portfolio constraint

$$D[< S_{\sim/o} >_{-/o} + D[\beta_{\sim/o} < H_{\sim/o} >_{-/o} + \alpha_{\sim/o} \ln \int (\Pi + \Delta_{\sim/o} x) dx] = 0$$

$$let\ \Delta_{-/o} = \frac{\partial f_{-/o}}{\partial x} \tag{16}$$

and assuming similar profit desired r rate for the portfolios though as we saw or could surmise this can be relaxed to dissimilar r r' & even made time dependent & beyond that this explored elsewhere (unpublished).

We interchange subscripts and superscripts merely for ease of rendering, & recycle the Lagrange multiplier symbols. Here D is the extremizations, the variation w.r.t. the appropriate averaging statistical weight or PDF generally distribution. We note that the Delta hedge is also European or toy or trial model dependent.

Eq.(16) is then identical to beginning with a maximum entropy or negentropy information theoretical extremizations where we constrain the say maximization with the Hamiltonian and then additionally include other information such as the work corresponding to Free energy, the portfolio composition of/with delta hedges this also in the extremum of zero change in entropy (no heat transfer) the work, whereas the converse that of minimization of free energy is the max entropy approach, and the method of E.T. Jaynes of adding proportionately additional observables or constriants the intermediate form or departure from one extreme. It is often useful to recall the connectivity of these varying points of view as they are the insight needed. In this case we simply summarize this and say that we add what we know or can measure or what we are interested in into the maximization and derive the PDF. We ask the reader to recall that this is subtly different from our previous intended maximum entropy with purely Hamiltonians. Now we have the additional p.o.v. of portfolios as observables, and these may or may not be made to correspond to additional sources of uncertainty, additional Hamiltonian components as by superposition. Again alternatively, the reader may wish to merely view this as max entropy with additional portfolio delta comprised observables.

The variation acts upon the distribution(s) as

$$D\int f_{-} \ln f_{-} = \int (\ln f_{-} + 1) D f_{-}$$

$$D\int f_{o} \ln f_{-} = \int (\ln f_{-} + 0) D f_{o}$$

$$D\int f_{-} \ln f = \int (\ln f + 0) D f_{-}$$

(17)

and so on for other configurations of averaging. Additionally one may utilize the variation to transform between ensembles within the integrations or rather the statistical averaging as we did above when we derived relationships between the full twiddle and naught partition functions etc.

The average of the Hamiltonian (full) w.r.t. the ensemble or distribution proceeds straight-forwardly with the variation acting on the distribution only inside the integral(s), arbitrary variations which we do not need to specify here. The portfolio is merely time dependent and therefore the variation of that term is zero. However the Delta hedge term from both toy model and European portfolio model goes as a partial of the distribution. We note that we can separate the distribution by integration by parts as before, the partial acting on the x stock etc. and the boundary term now therefore zero by variation:

$$D\ln\int(\Pi+\Delta_{\sim/o}x)dx = -\frac{1}{\int(\Pi+\Delta_{\sim/o}x)dx}\int 1.Df_{\sim/o}dx$$

.

(18)The integral in the denominator is independent of x but is time dependent; the variation obtains the negative part, the boundary terms of the integration by parts obtain zero variation, of the PDF under the integral.

Therefore the entropy maximization results in the least biased toy or 'naught' (European or twiddle) distribution function averaging based result

$$\int\{(\ln f_{\_}+1)+\tilde{\beta}(H_{full/\sim})-(\frac{\tilde{\alpha}}{\int(\Pi+\tilde{\Delta}x)dx})\}Df_{\_}dx=0$$

$$f_{\_}=e^{-\tilde{\beta}(H_{f/\sim})}e^{\tilde{\alpha}(\int(\Pi+\Delta x)dx)^{-1}}$$

(19)

with the normalization N=1/Z to be determined, and as the Hamiltonian is the leading observable we can set the Lagrange multiplier beta as 1/kT thermodynamically (which dynamically is differentiated and where an interval -(beta/2) <= tau <= (beta/2) becomes the evolution parameter 'time' which then corresponds to (after some gymnastics) to the inverse diffusion coefficient in the Matsubara Diffusion Schroedinger picture & PDE F-P additionally) or to the inverse variance (~ diff. coeff.) and determine alpha additionally by the identity(s)

$$-\frac{1}{Z}\frac{\partial Z}{\partial\tilde{\beta}}=\langle H\rangle$$

$$-\frac{1}{Z}\frac{\partial Z}{\partial\tilde{\alpha}}=\langle(-(\int(\Pi+\Delta x)dx)^{-1})\rangle$$

. (20)

It should be noted that the identity itself is a form proportional to the partial differential of the Free energy w.r.t. the Lagrange multiplier, as F~-lnZ with Z the partition function or integral of the distribution function(s) and is the inverse of the normalization N=1/Z.

Given the simple form of the variation and the Legendre transform and due precisely to the integrated time-dependent form of the portfolio relationship integral dual to alpha_~ , the analog of superposition, we have pre-separated the Lagrange multipliers in form; otherwise we would have to have led with an 'overall' multiplier and then multiplied each additional observable non-Hamiltonian with additional multipliers, making the disentanglement of these multipliers a chore. We can now attempt to glean the form of these (separated, one for the Hamiltonian, one for the portfolio observable) multipliers.

Writing the variation in this derived form, we can solve for alpha and beta explicitly. However first let us derive the least biased max entropy toy model distribution,

$$D < S >_{\sim} + D[\tilde{\beta}(< \tilde{H} >_{\sim})] - D[< 1 >] = 0$$

$$f_{\sim} = e^{-\tilde{\beta}\tilde{H}}$$

$$\tilde{\beta} = \tilde{\beta}(t)$$

$$-\frac{1}{\tilde{Z}}\frac{\partial \tilde{Z}}{\partial \tilde{\beta}} = \frac{1}{\tilde{Z}}\int \tilde{H} f_{\sim} dx \qquad (21)$$

to which we can add the 'free energy' of the portfolio, however the constraints included, the Hamiltonian, is the sufficient constraint for the derivation of the Schroedinger or Fokker-Planck solution PDF and is the derived result in Eq.(21).

The Bogoliubov inequality obtained the insertion of the full Hamiltonian into the toy model thermodynamic relation (twiddle), and this obtains by comparison with the max entropy derivation of Eq.(21) the following:

(21.b)

However it must be noted that the Bogoliubov inequality substitution of the full Hamiltonian into the thermodynamic expression for the free energy of (twiddle) the toy model is actually the expectation

$$< H >_{-} = \frac{1}{\tilde{Z}} \int (H_o + H_1) e^{-\tilde{\beta} \tilde{H}} = < H_o >_{\sim} + < H_1 >_{\sim}$$

$$< \tilde{H} >_{-} = \frac{1}{\tilde{Z}} \int (H_o + < H_1 >_o) e^{-\tilde{\beta} \tilde{H}} = < H_o >_{\sim} + << H_1 >_{\sim} >_o$$

(21.c)

Noting that the additional constraint of the probability, which summation or integration equality to one 1 eliminates the exponential e^-1 otherwise obtained from the lnf negentropy maximization. And where beta_~ is the Lagrange multiplier related to the inverse variance, this the diffusion coefficient of g_o_~(x,t) the Forward Fokker-Planck PDE, the thermodynamic form of the alternatively and equivalently statistical diffusion (inverse) variance or conditional second moment based Lagrange multiplier, and we shorthand this as we can shift from one to the other as by Einstein relations or the also afore-mentioned Matsubara diffusion-Schroedinger methods readily (partial beta gives the Hamiltonian acting on f_~/o, the Schroedinger diffusion form). Therefore we have an explicit form for the (individual, separable) Lagrange multipliers and the distribution:

<u>Continuing the derivation:</u>

We noted above that an immediate maximum entropy variation of the full Hamiltonian (also H_o and H_~) is possible, and is not complicated by inequalities of portfolios or Free energies, and yields for the American style derivative Hamiltonian

$$f = \frac{e^{-\beta H}}{Z}$$

$$H = H_o + H_1$$

$$Z = \int f dx$$

(23)

and similarly for the toy or trial model and naught or European style derivative/option we had

$$f_{\sim} = \frac{e^{-\tilde{\beta}\tilde{H}}}{Z_{\sim}}$$

$$f_o = \frac{e^{-\beta_o H_o}}{Z_o} \qquad (24)$$

and where we have been explicitly super or subscripting all variables, observables and multipliers to distinguish these straight-forward maximization results which we will had the opportunity to examine.

We note again that naught and twiddle, the toy/trial model and the European style model may coincide and/or differ by a constant or time dependent factor by choice!

A reasonable course of action given the above derived results of partition function etc. relations is to examine how the full, and mixed inequality, and toy and European are connected at the Hamiltonian model levels, and given these connections which we established via thermodynamics and method of inequalities relate full & trial or toy models of the same. This towards understanding how the departure from H_o or the H=H_o + H_1 contributes to our previous letters I-II's additional sources of uncertainty due to early (arbitrary) exercise.

We also examine the results above further. The 'beta' Lagrange multiplier(s) go as inverse variances. Say the PDE adds variance.times.2nd order partial operators, this shows as an overall inverse variance, or a partitionable series of inverse variances corresponding to the partition. We showed this above.

The simple log-normal Black-Scholes model of European options, i.e. the cases of constant variance of constant diffusion coefficient for the one-dimensional x,t x=lnS here S=stock price but these as y,t with y=x-rt goes as the European market variance.

If we add to this L=H operator by transformed simple form as we intended, some similar-in-form additional variance multiplying 2nd order partials, we obtain precisely an additional source of uncertainty, and where this corresponds to the operator component in H Hamiltonian as

$$H = H_o + H_1$$
$$= \frac{\sigma_o^2}{2}\frac{\partial^2}{\partial y^2} + \frac{\sigma_1^2}{2}\frac{\partial^2}{\partial y^2} \qquad (25)$$

where variances are generalize-able to non-constants or y,t dependent coefficients, and with the full Hamiltonian (operator) consisting of the Black-Scholes Hamiltonian (European style y,t operator H_o) plus the additional source of uncertainty operator H_1 whose variance (inverse Lagrange multiplier, alternatively diffusion coefficient) we surmise goes most simply as a diffusion coefficient multiplying an operator (sans drift say..) which we simply write as a 2nd order partial differential operator.

This then corresponds precisely to the statements in letters I-II that Eq.(7.b.,7.c.) has added to it a surmised or here derived from Gibbs-Bogoliubov inequality considerations and thermodynamics (!), the additional diffusion-like 2nd order PDE operator composed of partial derivatives.

This re summarized is obtained once the Hamiltonian Schroedinger form has (b^2(y,t)->sigma^2 simply, is the diffusion coefficient, whose inverse and yet piece-wise Lagrange multiplier is 'beta' by the by) had to it added the additional Hamiltonian component, which simply we cast in the same 2nd order form given our detailed transformations from drift+diffusion to diffusion only.

Equivalently the uncertainty due to early and arbitrary exercise has been looked at mathematically as variance-times-2nd order partial-derivatives, by superposition. And when this Hamiltonian picture is re-transformed back to the Fokker-Planck picture, this presumably results in additional PDE terms in the European Fokker-Planck operator (PDE), 'making it' an American style option PDE.

The PDE then is modified by the (as in papers I-II) additional uncertainty terms, these mathematically the additional 2nd order partial differential operators on par with the European style y,t 2nd order operator(s).

Specifically, the F-P equation of y,t European style becomes

$$\partial_t g = (\hat{H}_o + \hat{H}_1) g$$
$$\partial_t g = (\hat{L}_o + \hat{L}_1) g$$
$$\partial_t g = \frac{(\sigma_o{}^2 + \sigma_1^2)}{2} \partial_y^2 g(y,t \mid y',t') \qquad (26)$$

and when this is re transformed to f(S,t)

$$\partial_t f = r(f - S\partial_S f) - \frac{(\sigma_o{}^2 + \sigma_1^2)S^2}{2}\partial_S^2 f \qquad (27)$$

and where the following is possible:

a) An exact solution of the log-normal version of PDE Eq.(27) with constant diffusion coefficients by the integral of the 2-point function (Cox solution). Or a similar Black-Scholes approach to an exact solution in terms of Cumulative functions of a transformed Eq.(27) to the diffusion eqn. Eq.(26) form (Black-Scholes). See references. The innovation here being that with log-normal type or beyond, Eq.(27) solved, with t<T<T_o or with T the (arbitrary) exercise time, the solution & equation should or must coincide with the European style option solution & equation only when T->T_o the European option terminal preset exercise time, as we discussed in paper II.

b) Generalization of Eq.(27) to non-constant and non-log-normal processes...that is to mixtures of drift and diffusion non-lognormal processes, such as for example preferably to history following nonextensive Tsallis statistics for the SDE and PDE (market processes & uncertainties) and additionally the early exercise process (exercise uncertainties) as an additional nonextensive or extensive process (say similarly to the recently proposed Vega dependent process).

c) (after writing this manuscript) Comparing with a recent development that renders the American portfolio inequality to an equality which is due to the inclusion of the Vega process as a diffusion contribution to the Black-Scholes PDE. The Vega process the 2nd order partial w.r.t. the underlying of the derivative instrument (alternatively the partial derivative of the delta). The Vega process extends the European equality type of B-S PDE via the modified diffusion coefficient b^2=b^(f,x,t) to a PDE that also compensates for the exercise uncertainty in terms of the modification by the derivative! Recall that we performed such modifications in the above letter, with simplest resulting in additional sources b^2=const, and more accurately in accounting for feedback, the nonextensive &/or history following b^2=b^2(f_q,x,t) the q-parameterized form. We are not comparing directly here, merely stating that other active research is obtaining conclusions that parallel, that additional exercise uncertainty can be accounted for as additional sources of diffusion-like uncertainty.

Conclusions:

We have in this letter sought to examine the assumptions we have made relevant to the American style option in letters I-II, these briefly summarized as by the introduction & in Appendix A, that in addition to markets' drift+diffusion uncertainties & trends, additional drift+diffusion like uncertainty & trends due to the arbitrary exercise time of American style derivative options must be accounted for in any accurate American style options dynamics and model and then possible formula.

We had previously stated that these additional uncertainty terms come from a random in time boundary problem, this related to the optimal stopping time problem, and where the random in time boundary problem was the p.o.v. of letter I. And where we then sought to introduce these uncertainty sources straight-forwardly into stochastics and PDEs and as portfolio level 'compensation slack functions' that transformed inequality Portfolios to equality portfolios in letter II.

In this letter we approached the problem from an alternate and well established point of view...inequality portfolios, unknown full American style option distribution functions and PDEs these also inequalities if derived from inequality portfolios, from thermodynamics and informatics given Hamiltonians and free energies and inequality approaches as by Gibbs-Bogoliubov.

Particularly we claimed in this letter that due to the equality of ensemble averages of the full and twiddle 'trial' or 'toy' Hamiltonians, we can find and equate relations between the full American and known European derivatives by utilizing the twiddle or toy model.

And furthermore as the connexion between full Free energy and toy or twiddle free energy is an inequality, and this being made equal by various components' superposition, these also expressible as expansions and variations etc., and this on the average i.e. the Gibbs-Bogoliubov inequality, we can utilize this to force the upper bound equating of the full and toy, these both referenced to the European. We thereby 'eliminated' the inequalities per specific well-known procedures of averaging and by the Hamiltonian and free energy relations.

We made as few assumptions as possible, most importantly that the inequalities between portfolio components and free energy and toy model and full Hamiltonian components and free energy... could be removed in favor of equalities by well-known and proven Gibbs-Bogoliubov inequality procedures, these specifically methods of obtaining such equalities by straight forward procedures and proven thermodynamics relations.

We than showed that these equality results due to the definition of toy and reference 'naught' (European) Hamiltonians, resulted from adding components or additional Hamiltonian components by superposition.

We were able to, with sufficient rigor due importantly to this thermodynamics and Hamiltonians separate approach, to further connect the European and American style options problem at the state function level. We utilized the price PDF (detrended, transformed to the Schroedinger-like diffusion picture) as the component of Free energy and of entropy etc., and at the portfolio level.

We connected this to the free energy defined by Hamiltonian components... and therefore also connected to the Fokker-Planck PDE level by Hamiltonian operator to Fokker-Planck operator connection, a connection which we explicitly described albeit for the simple log-normal operator. Descriptions of L F-P and H Schroedinger type operators made it possible to utilize the proven thermodynamics in describing how and what the relations and underlying additional operator components must be such that Gibbs-Bogoliubov holds.

We also utilized and throughout (& I-II) reminded that PDEs and SDE descriptions are macro and micro equivalents. Also for clarity we stated that the equating of free energy with portfolios' components introduced the issue of the additional Hamiltonian components, the portion by superposition that added to the European Hamiltonian and that the choice of the extra term(s) then in each description made the equating of the portfolio components (& PDEs') to the sources of exercise uncertainty possible from this letter's thermodynamics p.o.v.

From Hamiltonian and maximization of entropy, equivalently minimization in information theory, we were able to derive several descriptions of European and toy model and American model distribution functions which follow the inequality G-B Gibbs Bogoliubov derived toy model ensemble averaging. These then supposedly describing American style derivative options.

Utilizing the definitions of Lagrange multipliers and diffusion coefficients ad Hamiltonian superposition, we then connected this present work and its results to the import of the previous letters on American style options I-II which then explicitly allowed us to describe via Hamiltonians the way the additional Hamiltonians' components and given their additional Lagrange multipliers introduced additional potential + kinetic terms to the Hamiltonians, and where we stated that our choice, diffusion only (kinetic, simplest) corresponded to the previous additional diffusion sources describing early exercise or arbitrary exercise sources of uncertainty in time and price, i.e. obtaining simple additional diffusion coefficients multiplying 2nd order partial differentials equivalently to II and as guided or motivated by I.

Having described the problem in these new points of view and well known terms of thermodynamics etc., we will return to the desired goal of deriving simple closed form, exact expressions of American style options for beyond-log-normal models (these exact solutions known in the literature), the log-normal market & constant diffusion of early exercise solved in Eq.(27) above and by expected results (a) in the same section, however a desired nonextensive accuracy by history-following processes in markets' SDEs and equivalently macroscopically the obtained nonextensive derivatives' PDEs, yet with history following early exercise uncertainty (read, stochastic) yet to be solved exactly.

We will do this in our upcoming letters, and also where we intend to re define with additional clarity solutions (general etc.) and connections such as exploiting the connection between Euro and toy and full to describe one PDF in terms of the other(s), and to shed further clarity and to make better connections to results obtained in I-II and our present letter.

Appendix A:

The problem of additional uncertainty due to arbitrary exercise time can be from one perspective modeled at the equivalent microscopic stochastics differential equation SDE and the macroscopic statistical evolution PDE.

The stochastic SDEs evolve as

$$\overset{market}{dx} = a(x,t)dt + b(x,t)dW$$

$$\overset{BS_o}{dx} = -rxdt + bdW$$

$$\overset{BS_A}{dx} = -rxdt + bdW + cdQ \qquad \text{(A.1)}$$

In Eq.(A.1) the underlying market SDE is the first equation, traditionally the deterministic drift term a is set to a(x,t)=a.x and b(x,t) the variance or diffusion coefficient related term is set to b(x,t)=v.x such that the log-normal process is assumed to model the market, this as recently shown not capable of modeling outliers and fat tails and superdiffusion and other stylized facts of markets unlike more fully descriptive say nonextensive statistics and stochastic processes.

However as we mentioned in the body of the letter we will work with log-normal processes for illustration of derivation and import and return to generalizations and issues of high accuracy in later work.

The 2nd SDE in Eq.(A.1) is the implied riskless return rate SDE obtained from the resulting Black-Scholes European model. Note the market drift a is replaced with r by the usual delta term in the portfolio (otherwise in the Ito c.o.v., not shown).

The 3rd SDE in Eq.(A.1) is the simplified <u>model</u> for additional sources of uncertainty not obtained from the market price of the underlying, the uncertainty obtained from considerations such as investor risk preference or aversion, profit motive, trend following, bias and sentiment and so on and these possibly pegged to the market(s) themselves. These are written as an additional diffusion-only term for simplicity, yet drift+diffusion terms can be added and compensated for in the B-S portfolio delta hedge or otherwise transformed and proportioned between drift only, diffusion only (as above) or drift+diffusion generally.

And in support of this simplified model, consider the information theory derivation, re-examined in light of uncertainty or lack of knowledge, partial knowledge etc. due to the uncertainty of the arbitrariness of exercise.

The information measure of Shannon written as <I>=<ln.g> with g the forward PDE distribution playing the part of PDF yet with proportioned price, the minimization of the information measure given constraints of known processes or surmised processes (E.T.Jayne's maximum entropy or negative of entropy the Shannon information, a method of deriving successive approximations of increasingly descriptive models or theories of a system addition of terms and factors in the Legendre transform these weighted or made proportional (optimally) by Lagrange multipliers, the so-called theory of theory creation approach) is written as

$$\begin{aligned}
& let\ x = \ln S,\ b^2 = \sigma^2,\ c^2 = \sigma_{excer}^{\ 2},\ < x > = r(t-t') \\
& y = x - rt \\
& d < I > + d[\beta < (x - < x >)^2 > + \alpha < x^2 >] = 0 \\
& or \\
& d < I > + d[\beta < (y)^2 > + \alpha < y^2 >] = 0 \\
& < I > = \int g \ln g dx \\
& g = \frac{e^{-\beta(y)^2} e^{-\alpha y^2}}{Z} \\
& g = \frac{e^{-\frac{(x-rt)^2}{2b^2 t}} e^{-\frac{x^2}{2c^2 t}}}{Z}
\end{aligned} \qquad (A.2)$$

The function derived, containing two sources of uncertainty these observables on the average (expectation values) given the simplest log-normal process as in Black-Scholes underlying SDE market processes can be seen to obtain a g(x,t) log-normal (x=lnS, S=stock price say) which is solved by a two variance forward g Fokker-Planck PDE which transformed with time decay exponential pre-factors and transformation to the backwards form is a Black-Scholes PDE however now with additional uncertainty or risk due to arbitrariness in exercise time included as the second source of uncertainty.

From the microscopic SDE level, the two sources of uncertainty in superposition (addition, albeit 'ad hoc'...a microscopic simple model), one generally due to markets' prices of underlying security say stocks, and the other due to additional uncertainty in exercise risk, however these both applied at the effective r dependent Black-Scholes implied SDE of dx (!), results in the equivalent macroscopic forward PDE

$$
\begin{aligned}
&< dWdQ >= 0 \\
&< dW(t)dW'(t') >= \varsigma^2 \delta(t-t') \quad let\ \varsigma^2 = 1 \\
&\frac{\partial}{\partial t} g = -\frac{\partial}{\partial x}[rxg] + \frac{({b_{market}}^2 + c^2)}{2} \frac{\partial^2}{\partial x^2}[g]
\end{aligned}
\tag{A.3}
$$

where this is obtained straight-forwardly by application of the Ito formula to an arbitrary function stochastic c.o.v. variable transformation (Taylor series of dh(x) and then averaging <dx> & <dx^2>) and/or then by averaging over g as <..> and integration by parts as usual (see Gardiner or Risken) to obtain a Fokker-Planck PDE...and where b and c are as above market risk and arbitrary exercise risk.

And where these 'simple' approaches or models came from a consideration of the arbitrary or random in time (!) boundary conditions problem. We had derived an American style function equation from a Green's functions approach, transforming 2nd order in price PDE operators to 2nd order in time PDE operators in letter I, and then we had examined the temporal diffusion uncertainty in detail. It became apparent in that letter that this approach, the introduction of the second arbitrariness uncertainty as a diffusion-like process introducing an additional source of uncertainty, was a valid approach to try and possibly one which would result in good descriptiveness.

And where in the following letters II these issues were then approached from different perspectives, in one approach adding at the macro portfolio level slack functions

$$
\begin{aligned}
&d\Pi \geq df - \Delta dx \\
&\to d\Pi = df - \Delta dx - dk \\
&k(x,t;..) > 0
\end{aligned}
\tag{A.4}
$$

this approach analogous to adding slack variables in optimization problems for inequality equations in linear programming problems. We merely generalized the linear variables-based approach to functions and functionals (if need be), and assuming that our procedure is/was valid, which validity was assumed was to be proven out by subsequent calculations and applications.

The introduction in the approach of letter II and Eq.(A.4) of slack function k(x,t)>0 upon variation (dk, df, dx etc.) resulted in the introduction of 1st and 2nd order conditional moments, and upon removing drifts per the usual delta hedge procedure or even under assumptions of averaging means or even follow-through re transformations of (the resulting PDEs') additional and K derived drift and diffusion operators & moments' coefficients to diffusion only, resulted in the sought after or surmised 2nd order operators corresponding to arbitrary exercise motivated diffusions, uncertainties or randomness, and the introduced additional diffusion coefficients described in Eq.(A.1-A.3).

The letters I-II therefore additionally sought to establish how European style derivative modeling theories of Black-Scholes were to be modified or generalized to include additional uncertainty derived from the arbitrary exercise uncertainty (regardless of motivation or causation) pertaining to American style derivatives (options).

And where this then has led us to the approach of this letter, this approach one of thermodynamics and the Gibbs-Bogoliubov inequality. Hamiltonian operators in lieu of Fokker-Planck operators are derived, and the inequality obtained from describing a full unknown system's free energy in terms of a toy or trial model's free energy and the known European style portfolio free energy is utilized to recast the American style inequality portfolio and its free energy (an inequality in the portfolio relation) or to equate it with the Gibbs Bogoliubov inequality.

Again the simple examples from log-normal processes are utilized to determine Hamiltonians etc. for the illustration...more complex statistics' F-P PDEs and therefore Hamiltonian operators are left to future generalizations.

From the analysis and the superposition principle of Hamiltonians the additional terms leading from known (European) to known toy models to 'unknown' full are exploited to assert the suppositions of letters I-II, that the addition of Hamiltonian components which must meet the Gibbs Bogoliubov inequality correspond to the addition of additional sources of uncertainty and which or whose forms are 1st and 2nd order partial differential operators, that is terms corresponding to 1st and 2nd conditional moments -like functions which are drift and diffusion terms multiplying the L operator's 1st and 2nd order PDE partial differential operators. In detail and in the Fokker-Planck PDE and therefore also the equivalent microscopic SDE pictures of letters I-II.

Appendix B:

An alternative derivation of European style BS F-P PDE's SDE with additional source(s) of exercise uncertainty 'noise' is as follows. Add the noise to the PDE at the macroscopic level directly as

$$dx = adt + bdW + cdQ$$
$$dh = \frac{\partial h}{\partial x}(adt + bdW + cdQ) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}dx^2$$
$$dh = \frac{\partial h}{\partial x}(adt + bdW) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}dx^2 + c\frac{\partial h}{\partial x}dQ(t)$$
$$< dh >_W = < \frac{\partial h}{\partial x}adt + \frac{b^2}{2}\frac{\partial^2 h}{\partial x^2}dt >_W + c\frac{\partial h}{\partial x}(\approx [Q(t) - Q(t')]\frac{1}{\Delta t})dt$$
$$[Q(t) - Q(t')]\frac{1}{\Delta t} \sim dQ(t) \quad \therefore \text{rescaled}$$
$$\frac{d < h >_{g_o}}{dt} = \int (ag_o\frac{\partial h}{\partial x} + b^2 g_o\frac{1}{2}\frac{\partial^2 h}{\partial x^2} + (cg_o dQ)\frac{\partial h}{\partial x})$$
$$\frac{\partial g_o(x',t')}{\partial t'} + \frac{\partial (rx')g_o}{\partial x'} - \frac{1}{2}\frac{\partial^2 b^2(x',t')g_o}{\partial x'^2} + \tilde{c}^2 dQ(t') = 0$$

$$dx = adt + bdW + cdQ$$
$$cdQ = c.y.dt$$
$$\therefore$$
$$dx = (a + c.y)dt + bdW$$
$$dy = \tilde{c}dQ$$
$$dh = \frac{\partial h}{\partial x}([a + c.y]dt + bdW) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}dx^2 + \frac{1}{2}\frac{\partial^2 h}{\partial y^2}dy^2 \qquad \text{(B.1)}$$
$$\frac{\partial g_o(x',t')}{\partial t} + \frac{\partial [a + c.y]g_o}{\partial x} - \frac{1}{2}\frac{\partial^2 b^2 g_o}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \tilde{c}^2 g_o}{\partial y^2} = 0$$
$$\frac{\partial^2 \tilde{c}^2 g_o}{\partial y^2} = (\frac{\partial x}{\partial y})^2\frac{\partial^2 \tilde{c}^2 g_o}{\partial x^2}$$
$$\frac{\partial x}{\partial y} = c(t) +$$

This is dependent on averaging over each random component W or Q and the ensemble averaging g_o. After multiplying by dt, note that the first two equations are averaged and integrated by parts to separate g from h in the derivation of a PDE from an SDE, therefore redefine the differential and re-integrate the fwd Integral{dg.h} to Integral{g.dh} and re apply the operators on h the arbitrary function that takes an SDE to PDE. averages are per W and not Q so carry out the similar procedure as in a one-source of W and noting that averages in terms of W and Q are separate per the Wiener Gaussian correlated noise delta functions correlations we noted in Appendix A, and then redefine the components of the SDE such that the c term is added to the dx terms, obtaining results analogous to those discussed in Appendix

A. Conversely begin from dx and then Ito formula c.o.v. of dh and then merely average over W noise and not Q noise and obtain the PDE+noise.

$$dx = adt + bdW + cdQ$$

$$dh = \frac{\partial h}{\partial x}(adt + bdW + cdQ) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}dx^2$$

$$dh = \frac{\partial h}{\partial x}(adt + bdW) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}dx^2 + c\frac{\partial h}{\partial x}dQ(t)$$

$$< dh >_W = < \frac{\partial h}{\partial x}adt + \frac{b^2}{2}\frac{\partial^2 h}{\partial x^2}dt >_W + c\frac{\partial h}{\partial x}(\approx [Q(t) - Q(t')]\frac{1}{\Delta t})dt$$

$$[Q(t) - Q(t')]\frac{1}{\Delta t} \sim dQ(t) \qquad \therefore \text{rescaled}$$

$$\frac{d < h >_{g_o}}{dt} = \int (ag_o\frac{\partial h}{\partial x} + b^2 g_o\frac{1}{2}\frac{\partial^2 h}{\partial x^2} + (cg_o dQ)\frac{\partial h}{\partial x})$$

$$\frac{\partial g_o(x',t')}{\partial t'} + \frac{\partial (rx')g_o}{\partial x'} - \frac{1}{2}\frac{\partial^2 b^2(x',t')g_o}{\partial x'^2} + \hat{c}^2 dQ(t') = 0$$

(B.2)

Where this micro to macro depends on the noise addition and rescaling property. That is, a noise plus a noise is yet a noise, as is a difference of noises and as is a rescaling by some interval of that noise, which however we could have written explicitly.

To prove the point, let us decouple the SDE in Eq.(B.2)

$$dx = adt + bdW + cdQ$$
$$cdQ = c.y.dt$$
$$\therefore$$
$$dx = (a + c.y)dt + bdW$$
$$dy = \tilde{c}dQ$$
$$dh = \frac{\partial h}{\partial x}([a + c.y]dt + bdW) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}dx^2 + \frac{1}{2}\frac{\partial^2 h}{\partial y^2}dy^2$$
$$\frac{\partial g_o(x',t')}{\partial t} + \frac{\partial [a + c.y]g_o}{\partial x} - \frac{1}{2}\frac{\partial^2 b^2 g_o}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \tilde{c}^2 g_o}{\partial y^2} = 0$$
$$\frac{\partial^2 \tilde{c}^2 g_o}{\partial y^2} = (\frac{\partial x}{\partial y})^2 \frac{\partial^2 \tilde{c}^2 g_o}{\partial x^2}$$
$$\frac{\partial x}{\partial y} = a_o^{-1}[c_1(t) + c_o]^{-1}$$
$$\frac{\partial g_o(x',t')}{\partial t} + \frac{\partial [a(x,t) + a_1(x,t)]g_o}{\partial x} - \frac{(b^2 + \tilde{c}^2)}{2}\frac{\partial^2 g_o}{\partial x^2} = 0 \quad \text{(B.3)}$$

Where the decomposition of the 2nd stochastic source can be made with an additional drift to counter or remove the induced drift of coupling/decoupling. Alternatively a can be redefined.

In summary, we have tried to show that with proper rescaling and even from the simplest considerations, noise can be added from the macro PDE scale and micro SDE scale to an already diffusion or stochastic trajectory, resulting in additional variance or diffusion coefficient contributions, and especially when one considers the drift removal by delta hedging which replaces a with r.

Appendix C:

Alternative formulation I:

The concept of additional uncertainty which must be taken into account to properly describe American style derivatives options suggests additional, alternative formulations. We investigate one here.

The concept is simple enough...the American style derivative is assumed to randomly vary about the deterministic European style derivative, this due to precisely the uncertainty due to arbitrary exercise which can occur from t_o to T_final of the 'equivalent' fixed exercise time European style option (say).

The Portfolio of functionals is then written as usual as the instrument option but this hedged by the uncertain asset or security or its converse interpretation, the underlying in combination with the insurance-like option, but this now based on the European style option as

$$\pi = f - \Delta f_o$$
$$d\pi = df - \Delta df_o$$
$$d\pi = r\pi_o dt$$
$$df_o = c(f_o,t)dt + b(f_o,t)dW(t)$$
$$f_o = f_{Euro}(x,t)$$
$$df_o = c.f_o.dt + b(f_o,t)dW(t)$$
$$\frac{\partial f}{\partial t} = r(f - \frac{\partial f}{\partial f_o} f_o) + \frac{b^2(f_o,t)}{2}\frac{\partial^2 f}{\partial f_o^2}$$
$$b^2(f_o,t) = \sigma_1^2 f_o^2 \qquad \text{(c.1)}$$

The American style option's evolution is now dependent on the underlying as usual however with the underlying now the European style option, comprised of a deterministic+random proportioned drift & diffusion, with the deterministic component referenced to the deterministic traditional Black-Scholes type of derivative option.

In fact we specialize the arbitrary nonlinear drift+diffusion coefficients SDE of Eq.(c.1) to the European drift as a linear drift

$$df_o = c.f_o.dt + b(f_o,t)dW(t) \qquad \text{(c.2)}$$

and where the Black_scholes type of F-P equation becomes

$$\frac{\partial f}{\partial t} = r(f - \frac{\partial f}{\partial f_o} f_o) - \frac{b^2(f_o,t)}{2}\frac{\partial^2 f}{\partial f_o^2}$$
$$b^2(f_o,t) = \sigma_1^2 f_o^2$$
$$f(f_o,t\,|\,t') = e^{r(t-t')} g(f_o,t\,|\,t') \qquad \text{(c.3)}$$

and we choose the linear drift and linear volatility of the log-normal process for simplicity as that has a known solution though here a functional. Additionally if we choose the r drift b^2=f^1-q form we have another exact closed form solution that of the Tsallis power-law statistics though here also a functional.

Several points:

1) The solution of the American style option is then as usual, the Black-Scholes style solution for log-normal processes and as by cumulant functions (see references), yet with f_o playing the part of stock price or underlying stochastic asset or security etc.
2) The expression for the American style derivative can be seen to be a functional of the European style option. Moreover, the expectation or average, the 1st moment or mean or 'drift' (beyond that the 2nd moment and from conditional averaging) of the SDE Eq.(c.2) and PDE Eq.(c.3) is <f_o>=f_o the European style option price function.
3) The integral solution is based on the expected price of say a call function price calculation this C(T)=Ex[max{f_o(t_f=T) - K,0}] at the 'terminal' time of exercise, which is discounted in time to the present moment to obtain (note as we are dealing with 2-point functions to derive these solutions, care must be made in defining t,t'):

$$g(f_o,t') = \int g(f_o(t),t)g(f_o,t\,|\,f_o',t')df_o$$

$$f(f_o,t') = e^{-r(T-t')}\int_K^{\infty}(f_o(T)-K)g(f_o,t\,|\,f_o',t')df_o \qquad \text{(c.4)}$$

the integral now finite-infinite in boundaries the result is comprised of cumulant functions, obtaning the Black-Scholes solution although here generally by convolution with 2-point transition-conditional distribution functions, now depending on the underlying European option price.

4) the functional expression for the American style option depending on the underlying f_o, its PDE can be transformed to expressions depending on the (2nd) underlying of the (1st) underlying, the stock price (say..).

$$\frac{\partial g}{\partial t} = -rS\frac{\partial g}{\partial S} - \frac{g^{1-q}(S,t)}{2}\frac{\partial g}{\partial S^2}$$
$$d[y = Se^{rt}] = e()dS + Sre()dt = bdW$$
$$dx = -rdt + bdW$$
$$d[x+rt] = bdW$$
$$dy = bdW$$
$$dS = uSdt + bSdW$$
$$< dS >= uSdt$$
$$df_o = u_1 f_o dt + b_1 f_o dW$$
$$< df_o >= u_1 f_o dt$$

$$\frac{\partial g}{\partial t} = -r\frac{\partial g}{\partial y} - \frac{\sigma_1^2}{2}\frac{\partial g}{\partial y}$$
$$(\frac{\partial y}{\partial S}) = (g_o(\frac{\partial S}{\partial g_o}))^{-1} \qquad \text{(c.5)}$$

and even this form is reasonable easily solved as the European g_o of f_o is known, and so is the partial w.r.t. the stock price.

5) The g of f, the fwd Fokker-Planck PDE functional PDF or the PDF price distribution function(al) can be derived directly from maximum entropy. Assume the removal of market drift for preferred r drift by the delta hedge as usual, and then also note the PDE is for an r-drift log-normal ('implied SDE', and then de-mean as in the body of the letter (above) for x=ln(f_o) and dx=-rt + bdW or for y=x-rt for dy=bdW with b the diffusion coefficient 'volatility'.

Then maximize for Gibbs-Boltzmann entropy to obtain

$$D[< S >] + D[\beta < y^2 >] + norm. = 0$$
$$f(S,t) = \frac{e^{-\frac{(\ln S - rt)^2}{2\sigma_1^2 t}}}{Z(t)}$$
$$Z = \sqrt{4\pi\sigma_1^2 t} \qquad \text{(c.6)}$$

however note that the secondary market SDEs for the primary underlying the stock (say..) and the second primary the option, for these simplest log-normal type of processes however for linear drift processes is

$$dS = uSdt + bSdW$$
$$< dS >= uSdt$$
$$df_o = u_1 f_o dt + b_1 f_o dW$$
$$< df_o >= u_1 f_o dt \quad \text{(c.6.b)}$$

where the expectation is at the SDE level. And as we are intending the secondary (American style) market option to fluctuate and evolve about the deterministic trend of, or the average that is the European style option price function. Again, we 'model' this by design as a linear drift, however it may become apparent that upon additional investigation that the model would be better described by a nonlinear drift say a 2nd order that reverts to the mean. However as Black-Scholes merely posited a log-normal linear drift stochastic process and hammered the model to an exact solution we feel free to posit a simple first look. Generalized stochastic processes may either wait or one can look at the suggestive nonextensive models we refer to here.

And which due to an interest in statistics which accurately capture the stylized facts of financial markets such as superdiffusion of price increments and the emergence of outliers or fat tailed power-law distributions, interests us in deriving the nonextensive entropy analog, obtaining upon maximizing <S_q> the nonextensive Tsallis entropy the following power-law distributions

$$D[< S_q >] + D[\beta < y^2 >] + norm. = 0$$
$$f(f_o, t; q) = \frac{[1 + \beta_1(t)(q-1)(\ln f_o - rt)^2]^{\frac{1}{1-q}}}{Z(t;q)}$$
$$f_o(S, t; q) = \frac{[1 + \beta_o(t)(q-1)(\ln S - rt)^2]^{\frac{1}{1-q}}}{Z_o(t;q)} \quad \text{(c.7)}$$

(q values may be decided to differ for underlying primary and underlying European secondary by the way) where we include the European nonextensive for log-normal (ad hoc) generalization and the American style nonextensive log-normal (functional SDE) process; and where the simplicity of the derivation of the PDF being due to a log-normal process which then is demeaned to y,t, and we used y interchangeably in the change.of.variables c.o.v. and where the simple demeaned PDE is then generalized to q-parameterized power-law q-PDFs whose PDEs go as

$$\frac{\partial f_o}{\partial t} = r(f_o - S\frac{\partial f_o}{\partial S}) - \frac{f_o^{\,1-q}(S,t;r)}{2}\frac{\partial f_o}{\partial S^2}$$

$$\frac{\partial f}{\partial t} = r(f - f_o\frac{\partial f}{\partial f_o}) - \frac{f^{1-q}(f_o,t;r,q)}{2}\frac{\partial}{\partial f_o^{\,2}} f(f_o,t;r,q) \quad (c.8)$$

and these the European and American nonextensive, ad-hoc generalized log-normal processes.

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