The Dakotas War: An Adventure in Geometry

In an unexpected display of brinkmanship, North and South Dakota have seceded from the Union and declared unconditional war on the United States! They intend to immediately begin construction of a nuclear power plant for the purpose of producing plutonium for a Dakota Bomb.

While geometry cannot help them with their political calculations, let's see if we can help them plan their defenses! Rapid City, Sioux Falls, Fargo and Belfield (famous for its *Superpumper™* gas station[1]) form a rectangle, $\overline{EFGH}$. The Dakotans intend to construct a triangle of military bases (with rapid-response troops and anti-aircraft guns that can also enfilade the highways) at Rapid City and on highways 29 and 94; that is, they will find J on $\overleftrightarrow{FG}$ and K on $\overleftrightarrow{GH}$ such that $\overline{EJK}$ is an equilateral triangle with the nuclear power plant at its center, equidistant from each base.

Paul Yiu has written extensively – a whole four lines – on this geometric construction. Like a lot of mathematicians, he overuses the letters A, B, C, D, so I changed his notation to ours[2]:

> *This construction did not come from a lucky insight. It was found by an analysis! Let $\overline{EF} = \overline{GH} = a, \overline{FG} = \overline{EH} = b$. If $\overline{FJ} = y, \overline{HK} = x$ and $\overline{EJK}$ is equilateral, then a calculation shows that $x = 2a - \sqrt{3}b$ and $y = 2b - \sqrt{3}a$. From these expressions of x and y, the above construction was devised.*

Isn't that amazing? Paul Yiu "devised" a construction based entirely on two algebra equations. Because of the "strength and power" of algebraic calculations, we can be rid of all geometry theorems and replace the entire geometric proof with a couple of algebra equations. *Woo hoo!*

> *D. E. Smith[3] (p. 95) explained that the teaching of constructions using ruler and compass serves several purposes: "it excites [students'] interest, it guards against slovenly figures that so often lead them to erroneous conclusions, it has genuine value for the future artisan, and it shows that geometry is something besides mere theory…" For all the strength and power of algebraic analysis, it is often impractical to carry out detailed constructions with paper and pencil, so much so that in many cases one is forced to settle for mere constructability… We focus on incorporating simple algebraic expressions into actual constructions using the Geometer's Sketchpad™.*

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[1] Paul Yiu plugged a commercial product – *Geometer's Sketchpad™* – in the abstract of a paper about elegance in mathematics, so why not mention *Superpumper™* here? www.forumgeom.fau.edu/FG2005volume5/FG200512index.html

[2] www.researchgate.net/profile/Victor_Aguilar4/publication/291333791_Volume_One_Geometry_without_Multiplication

[3] Smith, David Eugene. [1911] 2013. ***The Teaching of Geometry***. Los Angeles, CA: HardPress Publishing

After quoting Smith (co-author of Wentworth's *Plane Geometry*) on the importance of geometric constructions using ruler and compass, Paul Yiu slides into extolling the "strength and power" of algebraic analysis and dismisses all geometric constructions as impractical, settling for "mere constructability," which his amazing four-line "proof" apparently represents. Then he smoothly transitions into his real job, which is selling *Geometer's Sketchpad™* for *McGraw-Hill*. Amazing. In his next performance, Paul Yiu will demonstrate how fast he can pedal a unicycle backwards!

Now let's do it right! (Consult *Geometry–Do* regarding any theorems you are unfamiliar with.)

Lemma
Let ρ be a right angle, σ be a straight angle and φ be the interior angle of an equilateral triangle. φ trisects σ and $\frac{1}{2}\varphi$ trisects ρ. The exterior angle of an equilateral triangle is $\rho + \frac{1}{2}\varphi$.

Dakota Defense Problem
Given a rectangle, $\overline{EFGH}$, find J on $\overleftrightarrow{FG}$ and K on $\overleftrightarrow{GH}$ such that $\overline{EJK}$ is an equilateral triangle.

> ***Solution***
> Build an equilateral triangle on $\overline{GH}$ with its apex, M, on the same side of $\overleftrightarrow{GH}$ as E and F. Let $J = \overrightarrow{EM} \cap \overrightarrow{GF}$. Build an equilateral triangle $\overline{EJK}$; observe that K is on $\overrightarrow{GH}$.
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> ***Proof***
> We must show that K is on $\overrightarrow{GH}$. Let $K = \overrightarrow{EN} \cap \overrightarrow{GH}$ with N the apex of an equilateral triangle built on $\overline{FG}$ on the same side of $\overleftrightarrow{FG}$ as E and H. Now, we must show that $\overline{EJK}$ is equilateral. By the centerline theorem, M is on the mediator of $\overline{GH}$ and so, by the transversal corollary and the triangle frustum mid-segment theorem converse, M is the midpoint of $\overline{EJ}$. Analogously, N is the midpoint of $\overline{EK}$. $\overline{EF} = \overline{HG} = \overline{MG}$; and, by the lemma, $\angle EFN = \angle MGN$; also, $\overline{FN} = \overline{GN}$. Thus, by SAS, $\overline{EFN} \cong \overline{MGN}$, which holds the equalities $\angle FNE = \angle GNM$ and $\overline{NE} = \overline{NM}$. If the angle between these sides, $\angle MNE$, equals φ, then $\overline{MNE}$ is equilateral. $\angle MNE = \angle GNF + \angle FNE - \angle GNM = \angle GNF = \varphi$. By the medial triangle theorem, $\overline{MNE}$ equilateral implies that $\overline{EJK}$ is equilateral. ■

This is orange belt; proof that it works for a parallelogram is black belt and is left as an exercise.

When Paul Yiu states this problem, he claims that the equilateral triangle is "inside the rectangle," which is clearly not always true. The triangle is inside a square and it is inside a few rectangles that are almost square; but, contra Paul Yiu, "inside" is not generally true. But Paul Yiu just found this problem in somebody else's textbook and, by happenstance, they had drawn it with the triangle inside the rectangle. It is easy to leap to conclusions about results that you are stealing!

1. Paul Yiu claims to have "devised" the solution to this problem entirely with two algebra equations. This is not true. He just smeared some algebra on top of someone else's work.

2. Paul Yiu claims that his solution always inscribes the equilateral triangle inside the rectangle. This is not true. The triangle is only inside rectangles that are almost square.

These are the types of mistakes that happen when one smears some algebra on top of a geometry theorem that one just finds on the internet. Paul Yiu found the algebraic lengths of a couple of segments; he did not prove anything. The person he was stealing from happened to draw a figure with the triangle inside the rectangle, so Yiu leaped to the conclusion that it always is. There is a reason why geometers prove theorems; it is so we are sure that we know what we are doing.

There are a lot of people in America who are afraid of geometry and their cowardice in the face of a subject that they do not understand drives them to attempt to replace geometry with algebra. To save geometry, we must shame these people by demonstrating that they do not understand geometry *or* algebra. American high schools will soon abandon geometry in favor of – *God forbid!* – statistics; it needs to be saved from Paul Yiu, Agostino Prástaro, *et. al.*

Admittedly, the prospect of the Dakotans – all 1.6 million of them – declaring unconditional war on the United States is a bit absurd. Other than beer prices being too high, I'm not even sure what their grievances are. Nothing that we can't settle over a few cases of Peppermint Schnapps!

However, the Dakota Defense really is of interest to military cadets. Soldiers are sometimes tasked with building something that they know will be targeted by the enemy – say, a munitions dump – in the middle of open farmland that paved roads have cut into rectangles. They know:

1. Their bases must be on paved roads so they can quickly move to confront enemy infantry approaching from anywhere, and so they can enfilade the roads to hit enemy vehicles.

2. Enemy aircraft are best met by anti-aircraft guns at the vertices of an equilateral triangle.

Helping the U.S. military fight more effectively is also why my textbook, *Geometry–Do*, emphasizes machine gun emplacement. Russia teaches real geometry in their high schools, not that bogus *Common Core* drivel, and we must too if we are going to fight them. It is ridiculous that American military officers go into battle without a scientific approach to laying ambushes.

REFERENCES

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