A Response function of Merton model and Kinetic Ising model

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Abstract

We study contagious defaults of banks by applying a voting model. The network of the banks are created by the relation, lending and borrowing among banks. We introduce the response function from Merton model. Using this response function we calculate the probability of default (PD) which includes not only changes of asset values but also the effects of connected banks' defaults using the mean field approximation. If we approximate the normal distribution which Merton model uses by tanh function, we can obtain the kinetic Ising model which represents phase transition. The asset volatility plays the role of temperature. In the low temperature limit, the model becomes the threshold model. We calculate PD which shows an effect of the situations around the bank as the additional PD using the self consistent equation.

I. INTRODUCTION

Human beings estimate public perception by observing the actions of other individuals, following which they exercise a choice similar to that of others. This phenomenon is also considered as social learning or imitation and studied several fields [1]. It is usually sensible to do what other people are doing. Hence, collective herding behavior is assumed to be the result of a rational choice according to public perception. In ordinary situations this is the correct strategy and sometimes erroneous decisions like the beauty contest of Keynes [2]. As a macro phenomenon, large social movement is the absence of central control or public communications. A well-known example of erroneous decisions is the bank run on the Toyokawa Credit Union in 1973. The incident was caused by a false rumor, the process of which was analyzed in detail by [3] and [4]. These phenomenon is known as an example of information cascade [5].

Herding behavior is represented as the response function. Threshold rules have been derived for a variety of relevant theoretical scenarios as the influence response function. Some empirical and experimental evidence has confirmed the assumptions that individuals follow threshold rules when making decisions in the presence of social influence. This rule posits that individuals will switch between two choices only when a sufficient number of other persons have adopted the choice. We have studied the voting model including herders. The model was introduced to explain the information cascade. We refer to herders such as the threshold rule as digital herders [6]. From our experiments, we observed that human beings exhibit behavior between that of digital and analog herders [7], [8]. Analog herders vote for each candidate with probabilities that are proportional to candidates' votes [9]. The analog herder has weaker herding power than the digital herder.

Bank defaults are contagious. The failure of single bank can be spread through financial networks. Over the past years after great recession in 2008, many researchers in various fields have been addressing the question to how to prevent financial contagion. Some of them studied especially inter-bank networks where banks lend to and borrow from each other with the threshold rule [10], [11], [12]. On randomly connected networks, a small fraction of initial activated when the network is not too sparse or too dense, a phase transition can be found. It is called the cascade window. The noise is only first fraction and the models are the deterministic model. The model with noise which do not have the cascade window was studied in [13].

The relations between borrowers and lenders play important role in the contagions. The behavior of banks is similar to the herder. The situation affects the status of the banks and the persons. The relation is represented by the response function in our voting model. We extend the voting model and use Merton model as a response function to apply to the contagious defaults. In this case we can introduce the network between the banks and the change of the asset price naturally. The latter is presented by the correlations of assets [14]. We show the relation to kinetic Ising model which represent the phase transition and the asset volatility plays the role of temperature [15], [16].

The remainder of this paper is organized as follows. In section 2, we introduce our voting model and mathematically define the herders. In section 3, we construct the response function using Merton model. In section 4, we show the relation to kinetic Ising model. In section 5 we calculate the probability of default affected by the situations around the bank. Finally, the conclusions are presented in the last section.

II. MODEL

Here we consider a voting model. We model the voting behavior of two candidates, C_{-1} and C_1, at time t, and C_{-1} and C_1 have $c_{-1}(t)$ and $c_1(t)$ votes, respectively. In each time step, one voter votes for one candidate, which means that the voting is sequential. Hence, at time t, the t-th voter votes, after which the total number of votes is t. Voters are allowed to see r previous votes for each candidate; thus, they are aware of public perception. Here r is a constant number.

A voter's vote is based on the number of previous r votes. We call these voters herders. Here the voter refers to the latest r votes. In this paper we consider the network, the lattice case only [17]. Therefore, at time t, r previous votes are the number of votes for C_{-1} and C_1, which is represented by $c^r_{-1}(t)$ and $c^r_1(t)$, respectively. Hence, $c^r_{-1}(t) + c^r_1(t) = r$ holds. If $r > t$, voters can see t previous votes for each candidate. In the limit $r \to \infty$, voters can see all previous votes. We define the number of all previous votes for C_{-1} and C_1 as $c^\infty_{-1}(t) \equiv c_{-1}(t)$ and $c^\infty_1(t) \equiv c_1(t)$. Here we specify r to be constant. We define $c(t)^r_1/r = 1 - c(t)^r_{-1}/r = Z(t)$.

A herder's behavior is defined by a response function. We will lead the response function in next section using Merton model. In the voting model the response function is defined by the function $F(Z)$. We have considered the several symmetric function, digital herder $F(Z) = \theta(Z - 1/2)$ where θ is the Heaviside function, analog herder $F(Z) = Z$, and tanh type herder $\tanh(Z - 1/2)$. In this paper we consider the asymmetric function for the response function.

III. A RESPONSE FUNCTION OF MERTON MODEL

A. Balance Sheet

We constructed the voter's model in previous section. Here we apply the model to the contagion of defaults. The voters correspond to the banks. The bank's state is decided by around banks. $C_{-1}(C_1)$ is the status default (non-default) instead of the candidate for the voting model. The banks decide their status sequentially as the voters.

Here we lead the response function for banks using the balance sheet. On the asset side, bank i holds external risk assets, a_i, inter-bank assets, l_i, and safe assets, b_i. On the liability side, there are deposits, d_i, inter-bank liabilities, $\bar{l}_i$, and net worth, S_i. The balance condition of the bank is $A_i = a_i + l_i + b_i = D_i + S_i = d_i + \bar{l}_i + S_i$. Here we introduce the asset value A_i, and debt value D_i. Here these are present values, not book values.

Banks are connected each other by the relation, lending and borrowing among the banks. The existence of the relation is expressed as the arrow, from the borrower to the lender. The amount of bank j's borrowings from bank i is expressed as δ_{ij}. There are relations $l_i = \sum_j \delta_{ij}$ and $\bar{l}_i = \sum_j \delta_{ji}$.

The solvency condition of bank i is

$$A_i = a_i + l_i + b_i > D_i = d_i + \bar{l}_i. \tag{1}$$

It means that present value of the bank is larger the liability. If the present value of the bank is negative, excess debt, the status of the bank becomes default. We show the balance sheets of the banks and the contagion of defaults in Fig.1.

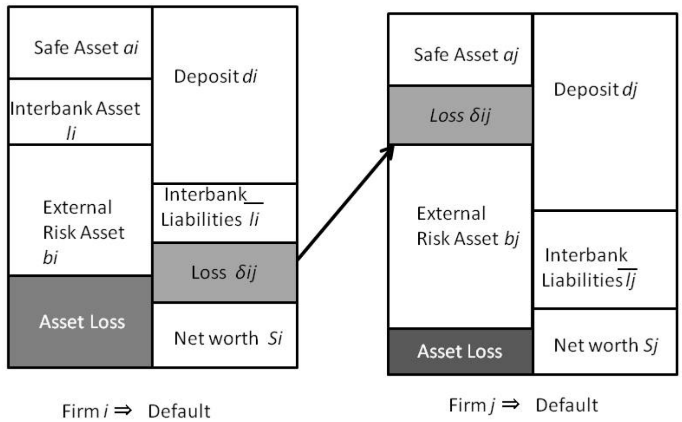


FIG. 1. The balance sheets of the banks. If the bank A is default because of the asset loss, the inter-bank liability from the bank B will be lost and the bank B be will be default contagiously.

B. Merton model

In this subsection we introduce the Merton model to calculate the probability of default (PD). We consider that stock price (market capitalization) is described as follows

$$dS_t = \mu_S S_t dt + \sigma_S S_t \sqrt{\epsilon}. \tag{2}$$

S_t is the stock price, which corresponds to the present net-worth at time t, μ_S is trend, and σ_S is the volatility of stock price. We omit the index of the firm i. Eq.(2) means the returns of stock price is log-normal distribution [18].

We consider the time series of whole balance sheet. The balance condition at time t is

$$A_t = D_t + S_t, \tag{3}$$

where A_t is asset, D_t is debt, and S_t is market capitalization at time t. Here we assume that the price of debt does not change, $D_t = D_0$. The initial condition of the balance sheet is $A_0 = S_0 + D_0$.

Using Eq.(2), we can obtain the stochastic differential equation for the asset A_t.

$$dA_t = \mu_A A_t dt + \sigma_A A_t \sqrt{\epsilon}, \tag{4}$$

where μ_A is the trend of asst, $\sigma_A$is the volatility of asset.

The default condition is $A_t < D_0$, as the solvency condition Eq.(1). Here we consider the default probability in the term T. We can obtain the probability of default of this bank at time T,

$$P(A_T < D_0) = N(\frac{\ln D_0 - (\ln A_0 + (\mu_A - \frac{\sigma_A^2}{2}))T}{\sigma_A \sqrt{T}}) = 1 - N(DD_0) = N(-DD_0). \quad (5)$$

$N(x)$ is the cumulative normal distribution and DD is

$$DD_0 = \frac{-\ln D_0 + \ln A_0 + (\mu_A - \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}}. \quad (6)$$

As DD_0 which becomes larger, the probability of default Eq.(5) decreases. Hence, DD_0 is "Distance to Default". We can observe μ_E and σ_E in the market stock price. The relation between μ_A and μ_E is

$$\mu_A = \frac{E_0}{A_0}\mu_E + (1 - \frac{E_0}{A_0})\mu_D, \quad (7)$$

where μ_D is the expected debt growth rate. The relation between σ_E and σ_A is

$$\sigma_E = \frac{A_0 N(d)}{E_0}\sigma_A, \quad (8)$$

where

$$d = \frac{-\ln D_0 + \ln A_0 + (\mu_A + \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}}. \quad (9)$$

Using Eq.(8) and Eq.(9) we can estimate σ_A and μ_A.

C. A response function with inter-bank liabilities

In this subsection we extend the previous subsection and calculate a response function which we use in the voting model from Merton model [19]. When the borrower j is default, the lender's inter-bank assets $\delta_{ij}(1 - R)$ are lost. Here we assume the recovery rate R is constant. DD in the condition that some borrowers defaulted, becomes,

$$DD(s) = \frac{-\ln D_0 + \ln[A_0 - \sum_{j=\text{default}}(1 - R)\delta_{ij}] + (\mu_A - \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}}. \quad (10)$$

We assume all inter bank assets are equal δ and the number connected banks is r. The total present value of the inter-bank assets is $l = \delta r = \sum_j \delta_{ij}$ and the number of the borrowers is r. In the voting model r is the number of referred voters.

We can rewrite

$$DD(s) = \frac{-\ln D_0 + \ln[A_0 - s(1-R)l/r] + (\mu_A - \frac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}}. \tag{11}$$

$DD(0) > 0$ by the solvency condition, but as s increases DD might be negative.

Here we change the valuable from s to Z, where $Z = s/r : 0 \leq Z \leq 1$.

$$DD(Z) = \frac{-\ln D_0 + \ln[A_0 - (1-R)lZ] + (\mu_A - \frac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}}, \tag{12}$$

and

$$\Phi(Z) = N(-DD(s)) = N(-DD(Z)). \tag{13}$$

Eq.(13) is the response function which included the status around the bank. When there is no default which the bank lent, DD is

$$DD(0) = \frac{-\ln D_0 + \ln[A_0] + (\mu_A - \frac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}}, \tag{14}$$

which is stand alone one which corresponds to Eq.(6).

Eq.(13) takes several shapes. If we set $\sigma_A \to 0$, the response function becomes steep and the Heaviside function. In the extreme case the response function becomes the threshold model. In this case the contagious defaults risk is stronger than the risk of the change of the asset values. The threshold is $\lfloor (A-D)/\delta \rfloor$ where $\lfloor x \rfloor$ are floor function. The model becomes threshold model in Watts model [10]. For example, $D/A = 0.9$, $\delta/A = 0.2$, $\sigma_A = 0.01$, $\Phi(i)$ becomes digital. If one of the borrowers is default, the default probability becomes 1, $\Phi(0) = 0$ and $\Phi(i) = 1$, $i \geq 1$.

We consider the sensitivity analysis to confirm the effects of parameters. When the asset volatility change $\sigma_A \to \sigma_A + \Delta_{\sigma_A}$ and $\sigma_A >> \Delta_{\sigma_A}$, the change of DD is

$$\Delta DD(Z) = -(T + \frac{DD(Z)}{\sigma_A})\Delta_{\sigma_A} \sim -(T + \frac{DD(1/2)}{\sigma_A})\Delta_{\sigma_A}. \tag{15}$$

If the volatility of asset increases, the decrease of DD does not depends of Z. Here we assume $l << A_0$. It is the parallel shift of DD.

When we consider the change of the inter-bank liability $l \to l + \Delta_l$ and $\Delta_l << l$,

$$\Delta DD(Z) = -\frac{(1-R)Z\Delta_l}{(A_0 - (1-R)lZ\sigma_A\sqrt{T}} \sim -\frac{(1-R)Z\Delta_l}{A_0\sqrt{T}}. \tag{16}$$

If the inter bank-liability increases, the decrease of DD is proportional to Z. As Z becomes large, the change of DD becomes large. It is the increasing of steepness of DD about Z.

IV. DYNAMICS OF THE MODEL

The state of firms is denoted by the vector $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_{r+1})$ with $\sigma_j = \pm 1$. $\sigma = 1(\sigma = -1)$ means the default(non-default). The i-th agents state at time t is $\sigma_i(t)$. Total number of agents is $(r+1)$. We consider the case where the updated agents is chosen by the rules. The ordering of update is from σ_1 to σ_{r+1}. After the update of σ_{r+1}, we update σ_1 and so on. We repeat this process. Hereafter, we define the updated state of the firm σ_j after n th time as $\sigma_j^{(n)}$. The initial condition is $\sigma_j^{(0)} = 0$. All banks are not default. Time t is the number of updated banks.

The update of a bank is described by response function, Eq.(13). At time t, the bank decides a state by the response function. The bank is connected with other banks by the relations, borrower and lenders. The bank has r borrowers and decides the state of the bank using the state of connected borrowers states. It means all banks are connected by the response function Eq.(13). Here we assume the balance sheets of all banks are the same for the simplicity.

The transition can be written

$$\begin{aligned}
\sigma_j = 1 \to -1 &: w_j(\boldsymbol{\sigma}) = \Phi(-DD_j), \\
\sigma_j = -1 \to 1 &: F_j w_j(\boldsymbol{\sigma}) = 1 - \Phi(DD_j) = \Phi(DD_j),
\end{aligned} \tag{17}$$

where $\Phi(-DD_j)$ is Eq.(13). Here we approximated normal distribution by logistic function,

$$\begin{aligned}
\Phi(-DD_j) &\sim \frac{1}{1+e^{\lambda_0 DD_j}} = \frac{-\ln D_0 + \ln[A_0 - \frac{(1-R)l}{2}] + \ln[1 - \frac{(1-R)l(\frac{s}{r}-\frac{1}{2})}{A_0 - \frac{(1-R)l}{2}}] + (\mu_A - \frac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}}, \\
&\sim \frac{1}{1 + \frac{1-p_{1/2}}{p_{1/2}} e^{[-\frac{(1-R)l\lambda_0}{(A_0-(1-R)l/2)\sigma_A\sqrt{T}} \frac{(s-\frac{r}{2})}{r}]}}, \\
&= \frac{1}{1 + \frac{1-p_{1/2}}{p_{1/2}} e^{[-\frac{(1-R)l\lambda_0}{2(A_0-(1-R)l/2)\sigma_A\sqrt{T}} \frac{(\hat{c}_1-\hat{c}_{-1})}{r}]}}, \\
&= \frac{1}{2}(1 + \tanh[\frac{(1-R)l\lambda_0}{4(A_0-(1-R)l/2)\sigma_A\sqrt{T}} \frac{(\hat{c}_1 - \hat{c}_{-1})}{r} + \frac{1}{2}\log\frac{p_{1/2}}{1-p_{1/2}}],
\end{aligned} \tag{18}$$

where

$$p_{1/2} = \frac{1}{1+e^{DD(1/2)}}, \tag{19}$$

and $\lambda_0 \sim 1.6$. Here we use the approximation $\log(1-x) \sim -x$ when $x << 1$ and $A_0 >> l$. We have changed the variables from s to $\hat{c}_1$ and $\hat{c}_{-1}$. $\hat{c}_{-1}(\hat{c}_1)$ is the number of defaults (non-defaults).

The transition can be written

$$
\begin{aligned}
&\sigma_j = 1 \to -1 : \\
&\quad w_j(\boldsymbol{\sigma}) = \frac{1}{2}(1 - \tanh[\frac{(1-R)l\lambda_0}{4(A_0-(1-R)l/2)\sigma_A\sqrt{T}}\frac{(\hat{c}_1-\hat{c}_{-1})}{r} + \frac{1}{2}\log\frac{p_{1/2}}{1-p_{1/2}}], \\
&\sigma_j = -1 \to 1 : \\
&\quad F_j w_j(\boldsymbol{\sigma}) = \frac{1}{2}(1 + \tanh[\frac{(1-R)l\lambda_0}{4(A_0-(1-R)l/2)\sigma_A\sqrt{T}}\frac{(\hat{c}_1-\hat{c}_{-1})}{r} + \frac{1}{2}\log\frac{p_{1/2}}{1-p_{1/2}}].
\end{aligned}
\tag{20}
$$

The process is nothing but the kinetic Ising model. (see Appendix A) The last term $\log p_{1/2}/(1-p_{1/2})$ corresponds to the outer field. The correspondence to the parameter for Ising model is

$$
\frac{(1-R)l\lambda_0}{4(A_0-(1-R)l/2)\sigma_A\sqrt{T}} = \beta J. \tag{21}
$$

The condition of no outer field is $DD(1/2) = 0$. We can obtain the condition

$$
A_0 - (1-R)l\frac{1}{2} = D_0 e^{-(\mu_A - \frac{\sigma_A^2}{2})T}. \tag{22}
$$

σ_A, volatility of asset, corresponds to the temperature in Ising model. When $\mu_A \sim 0$ and $\sigma_A^2 \sim 0$, the condition of the symmetric is $A_0 - D_0 = E_0 = (1-R)l/2$ which is discussed in [10] and [11] where l/E_0 is the threshold. These model are the low temperature limit of our model.

The mean field equation is

$$
\tanh[\frac{(1-R)l\lambda_0\hat{Z}}{4(A_0-(1-R)l/2)\sigma_A\sqrt{T}} + \frac{1}{2}\log\frac{p_{\frac{1}{2}}}{1-p_{\frac{1}{2}}}] = \hat{Z}, \tag{23}
$$

where $\hat{Z} = 2Z - 1$. The critical condition of the symmetric case is

$$
\sigma_{Ac} = \frac{(1-R)l\lambda_0}{4(A_0-(1-R)l/2)\sqrt{T}}. \tag{24}
$$

V. ADDITIONAL PROBABILITY OF DEFAULT

In the ordinal case the probability of the default of the bank is calculated $\Phi(-DD_i(0))$ which is stand alone PD. It corresponds to that there is no default in the banks which the bank i lent to. PD depends on the situation of the bank i. If the some of the banks which the bank i lent to are defaults, PD of bank i increases. We calculate the non conditional PD as $\bar{P} = \int \Phi(DD(Z))d\mu(Z)$ where $\mu(Z)$ is the measure of Z.

The difference to the stand alone PD, which is PD excluded the effects of the other banks, is defined as the additional PD [20],

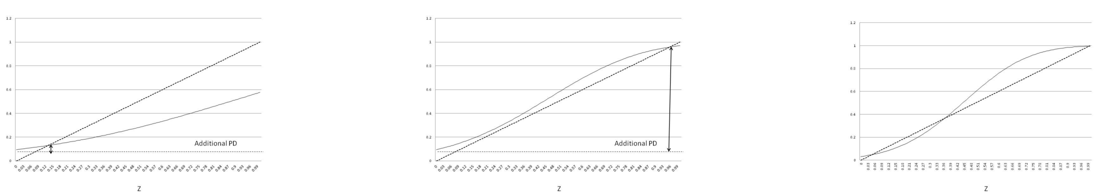
$$\Delta P = \bar{P} - \Phi(-DD(0)). \tag{25}$$

We calculate the additional PD using the mean field approximation,

$$\bar{P} = Z = \Phi(-DD(Z)). \tag{26}$$

We show the mean field equation in Fig 2. If there is only solution of the mean field equation, the intersection is the equilibrium solution in Fig 2 (a) and (b). Hence, the intersection becomes the equilibrium PD, $\bar{P}$. The distribution of Z is $\mu = \delta_{Z_1}$, where $Z_1 = \bar{P}$ and δ_x is the delta function. We can obtain the additional PD, $\Delta P = Z_1 - \Phi(-DD(0))$.

On the other hand,in Fig 2 (c) there is three solutions, both ends intersections are the solutions. The mid solution is not stable. The distribution of Z is $\mu = \alpha\delta_{Z_1} + \beta\delta_{Z_2}$, where Z_1 and Z_2 corresponds to the two stable solutions. We can obtain the additional PD, $\Delta P = \alpha Z_1 + \beta Z_2 - \Phi(-DD(0))$.



(a) (b) (c)

FIG. 2. The mean field equation and the solution of the equilibrium. (a) is small additional PD case, $D_0/A_0 = 0.91, l/A_0 = 0.1, \sigma_A = 0.07$ (b) is large additional PD case $D_0/A_0 = 0.91, l/A_0 = 0.2, \sigma_A = 0.07$ and (c) is the two solutions case $D_0/A_0 = 0.91, l/A_0 = 0.2, \sigma_A = 0.05$

In Fig. 3 we show the image of the two solutions case using the image of the physical potential. The ball comes from the left side, and there is no default in the initial condition. The solution oscillates between the two stable solutions Z_1 and Z_2 where $Z_1 < Z_2$. When

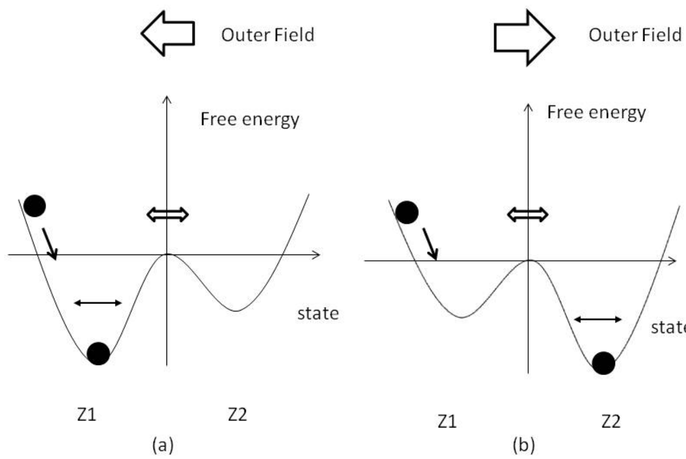


FIG. 3. Illustration of the equilibrium PD when there are two solutions, using the analogy of physical potential. The ball comes from the left side.

the outer field is the left (right) direction, the potential of the low (high) PD equilibrium Z_1 (Z_2) is deeper than the high (low) PD equilibrium in Fig. 3 (a) and (b).

When there is no outer field, symmetric case, one of the solution of Eq.(26) is $Z = 1/2$. When $d\Phi(-DD(Z)/dZ > 1$, there is three solutions. On the other hand, when $d\Phi(-DD(Z)/dZ < 1$, there is one solution. There is the phase transition in the limit $r \to \infty$ as Ising model. The condition of the critical asset volatility σ_{Ac} is

$$\sigma_{Ac} = \frac{\Phi'(0)l(1-R)}{(A-(1-R)l/2)\sqrt{T}}. \tag{27}$$

It is consistent with the critical condition of the Ising model Eq.(24), because $\Phi'(0) \sim \lambda_0/4$.

VI. CONCLUDING REMARKS

We considered contagious defaults of banks and applied a voting model to them. The network of the firms are created by the relation, lending and borrowing. We introduced the response function from Merton model. Using this response function we calculate the PD which includes not only the changes of the asset value but also the effects of other banks' defaults. The temperature corresponds to the asset volatility. When the asset volatility is small, the contingent default is effective for banks. On the other hand, the asset volatility is large, the change of asset price is effective.

In this paper we use the mean field approximation to calculate the PD. In general we have to do numerical simulations. If we use the random number including the correlations, we can simulate the correlation of the asset prices which several banks have.

Merton model uses the cumulative normal distribution. If we approximate the normal distribution by tanh function, we can obtain the kinetic Ising model. If there is no outer field, symmetric case, there is the phase transition. We show the additional PD which corresponds to the effects of the situations of the bank using the mean field approximation.

Appendix. Ising model

Here we consider the infinite range model. It is one of the most popular model in statistical physics which explains phase transition. In the model spins interact all other spins. Hamiltonian is

$$H(\sigma) = -\frac{J}{r+1}\sum_{i>j}\sigma_i\sigma_j - h\sum_{i=1}^{r+1}\sigma_i, \qquad \text{(Appendix 28)}$$

where $(r+1)$ is the number of spins, σ_i is the spin has the value ± 1, J is the parameter of interaction, and h is the outer field. Here we define average of spins, as an order parameter, $m = 1/(r+1)\sum\sigma_i$ which represents the phase transition.

In large $r \to \infty$ limit the self consistent equation is

$$m = \tanh\beta(Jm+h), \qquad \text{(Appendix 29)}$$

where $\beta = 1/k_B\hat{T}$. k_B is Bolzman constant and $\hat{T}$ is temperature. When infinite range model, we can obtain the strict solution form the self consistent equation Eq.(Appendix 29). When the symmetric case, under the transition temperature $\hat{T}_C$, in the low temperature range there are two solutions. The critical point is decided by the equation $\beta_c J = 1$. One of the solution is selected in the two stable solution. On the other hand, above $\hat{T}_c$, in the high temperature range, there are only one solution. This is the phase transition of Ising model. When there is the outer field h, the model becomes the asymmetric model and there is no phase transition.

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