Stochastic model of financial markets reproducing scaling and memory in volatility return intervals

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ABSTRACT

We investigate the volatility return intervals in the NYSE and FOREX markets. We explain previous empirical findings using a model based on the interacting agent hypothesis instead of the widely-used efficient market hypothesis. We derive macroscopic equations based on the microscopic herding interactions of agents and find that they are able to reproduce various stylized facts of different markets and different assets with the same set of model parameters. We show that the power-law properties and the scaling of return intervals and other financial variables have a similar origin and could be a result of a general class of non-linear stochastic differential equations derived from a master equation of an agent system that is coupled by herding interactions. Specifically, we find that this approach enables us to recover the volatility return interval statistics as well as volatility probability and spectral densities for the NYSE and FOREX markets, for different assets, and for different time-scales. We find also that the historical S&P500 monthly series exhibits the same volatility return interval properties recovered by our proposed model. Our statistical results suggest that human herding is so strong that it persists even when other evolving fluctuations perturbate the financial system.

Introduction

To estimate risk in a financial market it is essential that we understand the complex market dynamics involved.[1,2] Although our current understanding of financial fluctuations and the nature of microscopic market interactions remains limited and ambiguous,[3,4] as vast amounts of financial data have become more available we are now able to apply advanced methods of empirical analysis to gain greater insight into the market's complexity.[1,2,5,6]

Here we use a general agent-based stochastic model[7] and find that it is able to reproduce various statistical properties of a financial market.[8–12] We focus on volatility, which we define as fluctuations in the absolute returns, across a wide range of time-scales. Our task is to find the relationship between market volatility and market trading activity.[13,14] We demonstrate that our stochastic model allows us to understand absolute return intervals even when the values are extreme. We find that the statistical properties of return intervals are universal for a broad range of financial markets, from NYSE and FOREX. The model can reproduce these statistical properties by using the same set of parameters for varying time-scales, from high frequency data to monthly S&P500 index values across a 145-year period.[15] These results imply that the various power-law statistics of financial markets might be due to a non-linear stochasticity, which we incorporate into the herding-based model of financial markets.[16,17] Though the proposed model is designed to analyze statistical properties of volatility and the price of assets is not considered, the revealed bursting behavior extends our understanding of bubbles in financial markets[14,15] in general.

Methods

We use a modified version of the three-state agent-based model[7,18] to reproduce and explain the origin of the statistical properties of volatility return intervals.[8–10] The interplay between the endogenous dynamics of agents and exogenous noise is the primary mechanism responsible for the observed statistical properties. By exogenous noise we mean information flow or/and order flow fluctuations.

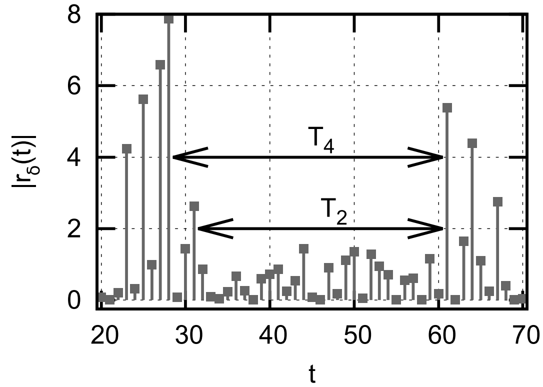

Figure 1. The definition of return intervals T_q between the volatilities of the price changes that are above a certain threshold q, measured in units of standard deviations of returns (not absolute returns). Here two values of threshold $q = 2$ and $q = 4$ are shown in the time series of absolute return.

Endogenous versus exogenous

The standard price model[19] and autoregressive conditional heteroskedasticity (ARCH) family of models[20,21] serve as phenomenological frameworks consistent with endogenous volatility and exogenous noise. For example, by analogy with ARCH family models we can assume that the log return $r_\delta(t) = \ln P(t) - \ln P(t-\delta)$ of the market price $P(t)$, defined at any moment t for a time interval δ can be modeled as a product of endogenous volatility $\sigma(t)$ and exogenous noise $\omega(t)$

$$r_\delta(t) = \sigma(t)\omega(t). \tag{1}$$

Here for the sake of simplicity we use a Gaussian noise $\omega(t)$, and volatility $\sigma(t)$ is assumed to be a linear function of the absolute endogenous log price $|p(t)| = |\ln \frac{P(t)}{P_f}|$

$$\sigma(t) = b_0(1 + a_0|p(t)|), \tag{2}$$

where $p(t)$ can be derived from the agent-based model (ABM) defining the ratio of market price $P(t)$ to fundamental price P_f.[7] Here b_0 serves as a normalization parameter, while a_0 determines the impact of endogenous dynamics on the observed time series. Our model, defined by Eqs. (1) and (2), comprises both the dynamic part described by $\sigma(t)$ and the stochastic part described by $\omega(t)$.

ABM

We use a version of the three-state agent-based herding model[7,18] to describe the endogenous dynamics of agents in the financial markets and to reproduce the statistical properties of volatility return intervals.[8–10]

The dynamics of agent population n_i under constraints $\sum_i n_i = 1$ are described by stochastic differential equations (SDEs) derived from the master equation with one-step transition $i \to j$ rates proposed by Kirman:[22]

$$\mu_{ij} = \sigma_{ij} + h_{ij} n_j, \tag{3}$$

where σ_{ij} describes the individualistic switching tendency, and h quantifies influence of peers (n_j). Note that a symmetric relation $h_{ij} = h_{ji}$ is usually assumed. A basic understanding of financial market dynamics allows us to make assumptions that simplify the model.

We first assume that the three states correspond to three trading strategies: fundamental (f), optimistic (o), and pessimistic (p). Fundamental traders assume that the price will be equal to a fundamental price P_f that is determined using market fundamentals.[23–27] Optimistic and pessimistic trading are two opposite approaches in the same chartist (c) trading strategy, i.e., optimists always buy and pessimists always sell. Mathematical forms of the excess demands, D_i, for both fundamental and chartist strategies are given by[24]

$$D_f = n_f \left[\ln P_f - \ln P(t)\right], \quad D_c = r_0(n_o - n_p) = r_0 n_c \xi, \tag{4}$$

where $P(t)$ is the current market price of an asset, r_0 the relative impact of chartists, and $\xi = \frac{n_o - n_p}{n_c}$ the average mood. Combining D_f and D_c, we obtain the log-price,[7,24]

$$p(t) = \ln \frac{P(t)}{P_f} = r_0 \frac{n_c}{n_f} \xi = r_0 \frac{1 - n_f}{n_f} \xi. \tag{5}$$

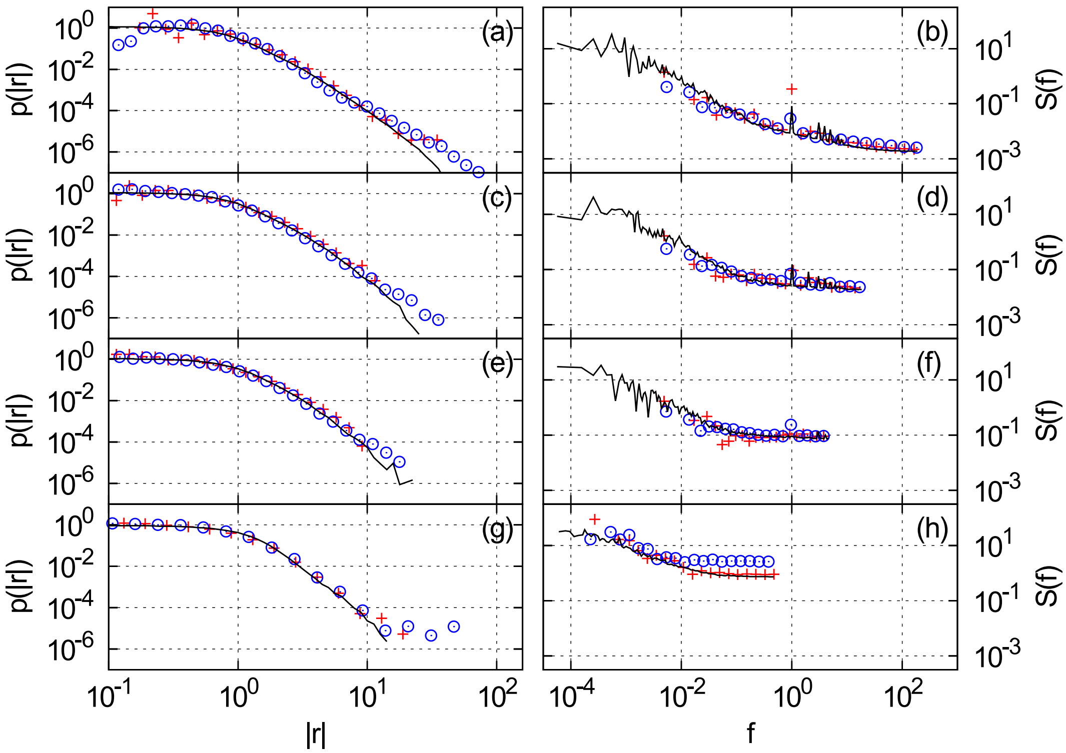

Figure 2. A comparison between the theoretically calculated (black lines) stationary PDFs (a, c, e and g) and PSDs (b, d, f and h) and the empirical results for NYSE stocks (circles) and FOREX exchange rates (pluses). (a) and (b) for time-scales $\Delta = 1/390$ trading day ; (c) and (d) - $\Delta = 1/39$ trading day ; (e) and (f) - $\Delta = 4/39$ trading day , frequencies in PSD graphics are given in 1/(1 trading day). Results for time scales $\Delta =$ one trading day of all considered assets from NYSE (circles) and of 10 exchange rates from FOREX (pluses) are in (g) and (h), where empirical series are from 1962 to 2014 year for NYSE series and from 1971 to 2014 year for FOREX. Model parameters are set as follows: $h = 0.3 * 10^{-8} s^{-1}$; $\delta = 3.69$ min.; $\varepsilon_{cf} = 1.1$; $\varepsilon_{fc} = 3$; $\varepsilon_{cc} = 3$; $H = 1000$; $a_0 = 1$; $a_\tau = 0.7$; $\alpha = 2$ for both NYSE and FOREX.

We next simplify the model by assuming that optimists and pessimists are high-frequency trend followers, i.e., chartists. Chartists trade among themselves H times more frequently than with fundamentalists. There is no genuine qualitative difference between optimists and pessimists in terms of herding interactions, and certain symmetric relationships are thus implied ($\sigma_{op} = \sigma_{po} = \sigma_{cc}$ and $h_{op} = Hh_{fc} = Hh$). Chartists share their attitude towards fundamental trading ($\sigma_{pf} = \sigma_{of} = \sigma_{cf}$) and fundamentalists are indifferent to arbitrary moods ($\sigma_{fp} = \sigma_{fo} = \sigma_{fc}/2$ and $h_{fp} = h_{fo} = h$). The assumption that fundamentalists are long-term traders and chartists short-term traders can be written as ($H \gg 1$, $\sigma_{cc} \gg \sigma_{cf}$ and $\sigma_{cc} \gg \sigma_{fc}$). Under these assumptions the dynamics is well approximated by two nearly independent SDEs[7,18] that resemble the original SDE from the two-state herding model,[22,24]

$$\mathrm{d}n_f = \frac{(1-n_f)\varepsilon_{cf} - n_f \varepsilon_{fc}}{\tau(n_f)}\mathrm{d}t + \sqrt{\frac{2n_f(1-n_f)}{\tau(n_f)}}\mathrm{d}W_f, \tag{6}$$

$$\mathrm{d}\xi = -\frac{2H\varepsilon_{cc}\xi}{\tau(n_f)}\mathrm{d}t + \sqrt{\frac{2H(1-\xi^2)}{\tau(n_f)}}\mathrm{d}W_\xi, \tag{7}$$

where $\tau(n_f)$ is the inter-event time, and W_f and W_ξ are independent Wiener processes. Note that in the above equations we scale model parameters, $\varepsilon_{cf} = \sigma_{cf}/h$, $\varepsilon_{fc} = \sigma_{fc}/h$, and $\varepsilon_{cc} = \sigma_{cc}/(Hh)$, as well as time $t_s = ht$ (omitting the subscript s in the equations).

We consider the inter-event time $\tau(n_f)$ a macroscopic feedback function, which can take the form

$$\frac{1}{\tau(n_f)} = \left(1 + a_\tau \left|\frac{1-n_f}{n_f}\right|\right)^\alpha. \tag{8}$$

This form is inspired by empirical analyses,[28–31] where the trading activity is proportional to the square of the absolute returns (thus $\alpha = 2$). This form depends on the long-term component of returns in the proposed model (see[32]) and $\frac{1}{\tau(n_f)}$

converges to unity when n_f approaches 1. The trading activity never reaches zero, and in non-volatile periods it fluctuates around some equilibrium value.

Equations (6)–(8) consitute the complete set for the macroscopic description of endogenous agent dynamics and together with Eq. (5) consitute a model of financial markets. Equation (6) written for the new variable $x = \frac{1-n_f}{n_f}$ in the region of high values of variable belongs to the class of non-linear SDEs and reproduces power-law statistics: probability density function (PDF) and power spectral density (PSD).[16,33]

We substitute the endogenous price $p(t)$, Eq. (5), calculated using Eqs. (6)–(8) for n_f and ξ, into Eqs. (1)–(2) to complete the model, which now includes the endogenous and exogenous fluctuations. It has been demonstrated[17] that the model now resembles versions of non-linear GARCH(1,1) models.[34,35]

The advantage of agent-based models over pure stochastic models is that their parameters are more closely related to real-world scenarios and real human behavior. We thus extend the model by first introducing relationships between exogenous and endogenous fluctuations and then adjusting their time scales. In particular, we relate the sign of the exogenous noise ω to the sign of the change in chartist mood during the return time window δ. This relation controls the impact of agent preference on market price dynamics and introduces the short-term return correlations observed in the real-world market, and this enables us to reproduce the empirical statistical properties with greater accuracy.

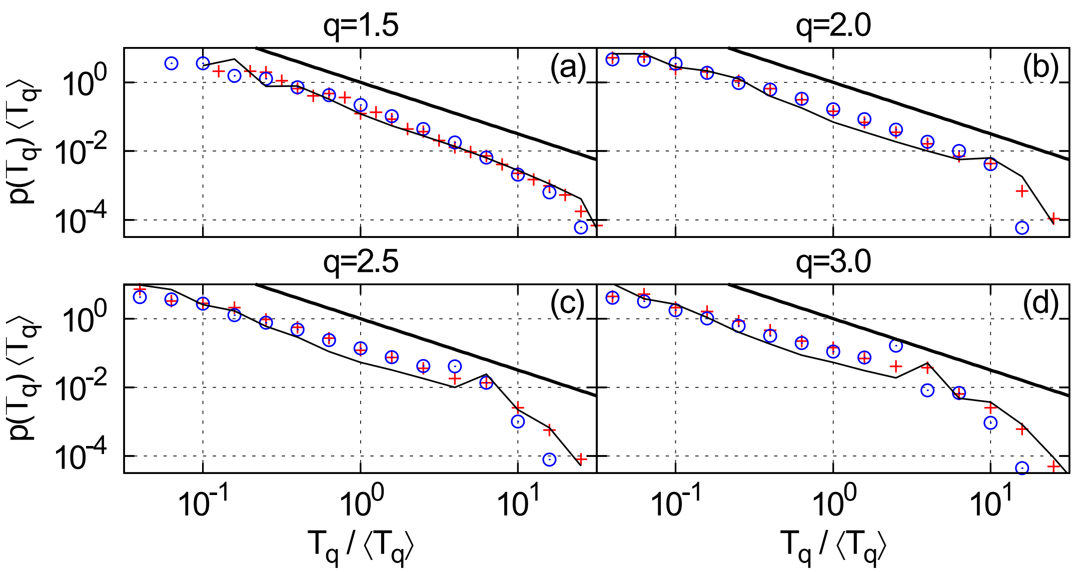

Figure 3. A comparison between the model (black lines) scaled unconditional PDFs of return intervals and the empirical PDFs calculated[8,9] from normalized series of NYSE stocks (circles) and FOREX USD/EUR exchange rate (pluses). All parameters of the model are the same as in previous figure, values of thresholds q are as follows: $1.5, 2.0, 2.5, 3.0$. The straight lines are shown to guide the eye showing a power-law with exponent $3/2$.

In the following we analyze the one minute, daily and monthly recorded time series. In numerical simulations we set 1/390th of a trading day as the smallest tick size δ, and individual returns are calculated between these ticks. We calculate the returns for long time periods Δ, e.g., one day, by summing up the consecutive short-time returns $r_\delta(t)$.

To account for the daily pattern observed in real data in NYSE and FOREX, we introduce a time dependence[7] into parameter b_0, i.e., $b_0(t) = b_0 \exp[-(\{t \mod 1\} - 0.5)^2/w^2] + 0.5$, where w quantifies the width of intra-day fluctuations. Although the model is designed to reproduce the power-law behavior of absolute returns PDF and PSD, it also reproduces data for larger time scales including the statistical features observed in volatility return intervals.

Results

In this study we analyze the empirically established statistical properties of volatility return intervals in financial markets[8–10] and use the same definition of this financial variable shown in Fig. 1.

For two absolute return threshold values $q = 2$ and $q = 4$ the return intervals are T_2 and T_4, respectively. They measure the time intervals between consecutive spikes of absolute returns that exceed threshold value q, measured in units of standard deviation of the returns in the time series of the specific asset.

PDF and PSD of absolute return

We test how well the model reproduces the empirical PDF and PSD of returns for NYSE stocks and FOREX exchange rates across a wide range of time intervals Δ that range from 1/390th to 1 trading day. We set the model parameters to be $\delta = 1/390$ day $= 3.69$ min., which is equivalent to 1 NYSE trading minute, $\varepsilon_{cf} = 1.1$ and $\varepsilon_{fc} = 3$, which define the anti-symmetric distribution of n_f, $\varepsilon_{cc} = 3$, which ensures the symmetric distribution of ξ, $H = 1000$ which adjusts the PSDs of the empirical and model time series, $a_0 = 1$ and $a_\tau = 0.7$, which are empirical parameters defining the sensitivity of market returns and trading activity to the populations of agent states, $\alpha = 2$, which is selected based on the empirical analyses[28–31] and our numerical simulations confirm this choice as well, and $h = 0.310^{-8} s^{-1}$, which is the main time-scale parameter that adjusts the model to fit the real time-scale. All the parameter values are kept constant throughout the analysis that follows.

Figures 2(a)–2(f) compare high frequency NYSE and FOREX empirical data with the results of the model. The data comprise a set of 26 stocks traded for 27 months from January 2005 and the USD/EUR exchange rate during a 10-year period beginning in 2000, and the empirical return series are normalized using return standard deviation σ_Δ. Figures 2(a)–2(f) show that the model results are in a good agreement with the high frequency empirical PDFs and PSDs.

Return intervals of high frequency return series

Our goal now is to explain, using our model, the statistical properties of the return intervals of both stocks and currencies.[8,9] Figure 3 compares the unconditional PDFs of the model with the PDF obtained for 1/390th trading day returns of NYSE stocks and USD/EUR exchange on FOREX, and Fig. 4 compares the conditional distribution functions. These results support the model showing that it successfully reproduces both unconditional and conditional distribution functions. When q values are comparable with the returns from their power-law part of PDF, $q > 1.5$, the power-law behavior of return intervals prevails $P(T_q) \sim T_q^{-3/2}$. Notice that scaled unconditional PDFs of T_q empirical as well as model given in Figure 3 are nearly the same for each value of q. We do observe this power-law behavior with exponent $3/2$ in the model even when we simplify it by replacing the whole model by stochastic dynamics of $x = \frac{1-n_f}{n_f}$ defined in Eq. (6) and the other noises are switched off. The cutoff of this power-law behavior for high values of T_q appears when other noises are switched on again. Our numerical simulations of the model show that for values of q, comparable with returns in very tail of their power-law PDF, the exogenous noise in Eq. (1) is responsible for the deviations from $3/2$ law, when ξ as well as intra-day trading activity dynamics force the scaled PDF back to a power-law $3/2$ behavior. Such impact of the exogenous noise increases with higher values of time window Δ,

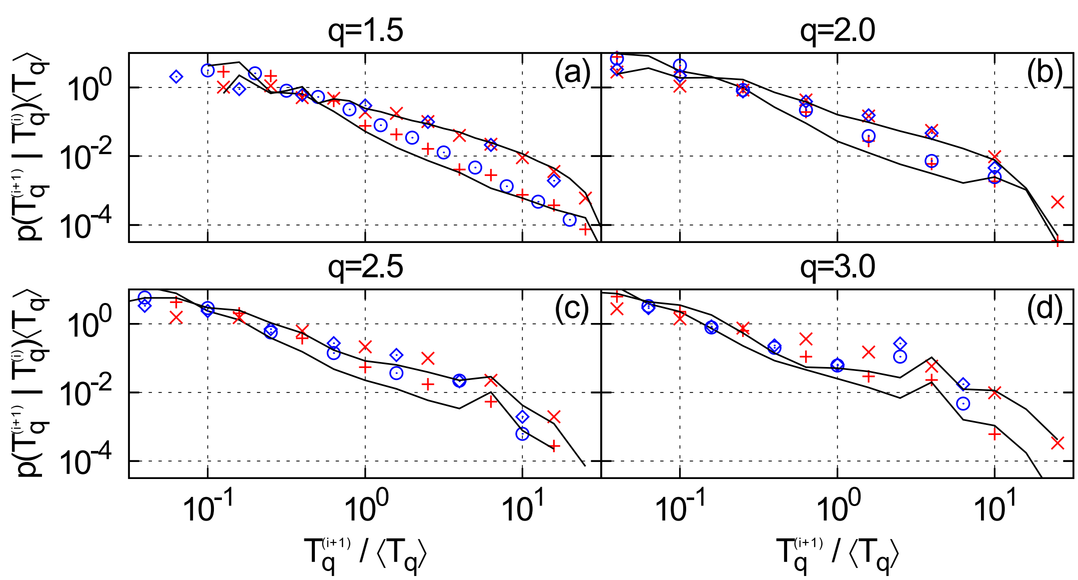

Figure 4. A comparison between the model (black lines) scaled conditional PDFs of return intervals, $P(T_q^{i+1} \mid T_q^i)$, and the empirical PDFs calculated from normalized series of NYSE stocks (circles and diamonds) and FOREX USD/EUR exchange rate (pluses and crosses). Conditional PDFs are calculated with the same algorithm as in ref.,[8,9] $T_q^i \le Q_1$ lower PDFs and $T_q^i \ge Q_8$ upper PDFs, where Q_1 and Q_8 are $1/8$ and $7/8$ quantiles of T sequence accordingly. Al parameters of the model are the same as in previous figures, values of thresholds q: $1.5, 2.0, 2.5, 3.0$.

For the threshold value $q = 1.5$, when $T_q^i \le Q_1$ and $T_q^i \ge Q_8$ the conditional distribution functions $P(T_q^{i+1} \mid T_q^i)$ are clearly different, indicating that there is a memory effect. Here i is the index in the consecutive T_q sequence, Q_1 the 1/8th quantile and Q_8 the 7/8th quantile of T_q series. When threshold values are higher the conditional PDFs become closer and might overlap. Our numerical modelings confirm that the necessary condition for this memory effect is the presence of long term dynamics, Eq. (6), and exogenous noise, Eq. (1). The speculative dynamics ξ and intra-day seasonality contribute to the dynamic behavior of the system, the persistence of a 3/2 power-law, and the memory effects. Note that all noises defined by the model are reflected in the PDFs of the volatility return intervals.

Our results support the empirical finding[8–10] that the PDF of the return intervals can be scaled to the same form common for different thresholds q. Note that the difference in scaling exponent between what we obtained $(3/2)$ and that obtained (2) in previous research is related to the use of different procedures for the return normalization, which in turn leads to different threshold choices. The thresholds used in previous papers are considerably lower than the ones we use in our model simulations and are outside the power-law portion of the return PDF. Because the contribution of the main SDE in Eq. (6) prevails over other noises only in the power-law portion of the return PDF, we choose higher values for threshold q and also show the deviation from the $3/2$ law for $q = 1.5$. This lowest value, $q = 1.5$, demonstrates the transition to the regime in which the return intervals are extremely short and the dynamic complexity of the signal extremely high. We cannot consider the high frequency fluctuations in this regime as caused by a one-dimensional stochastic process because other noises are also contributing. Thus the exponent of the return intervals tends to values higher than $3/2$. The empirical studies of return interval statistics described in Refs.[11,12,36] demonstrate the transition from a $3/2$ power-law to the exponential distribution of the unconditional PDF. Note that the authors of these studies also select lower values for the thresholds.

Return intervals of daily return series

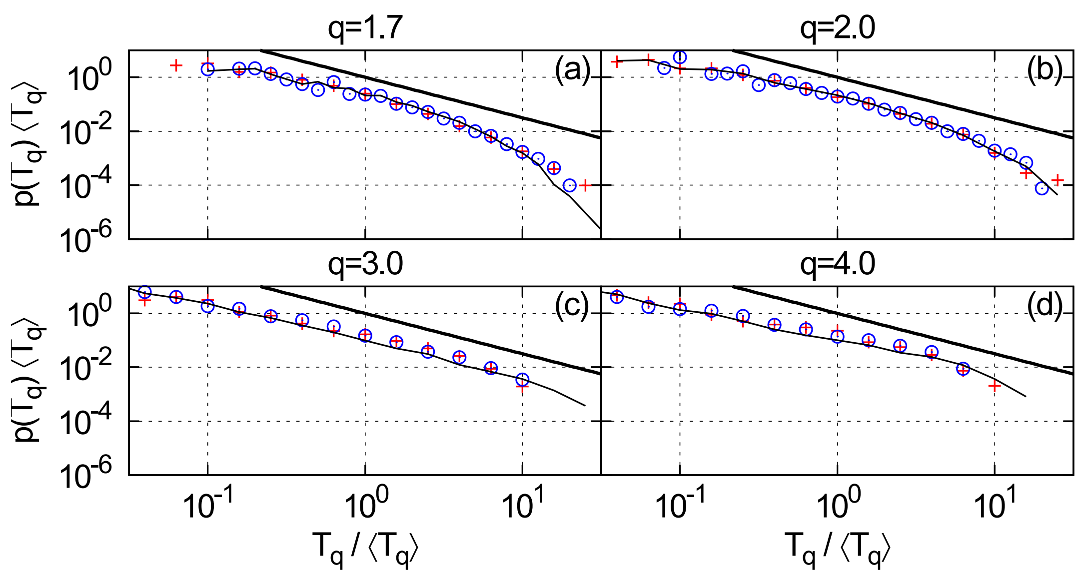

Figure 5. A comparison between the model (black lines) unconditional PDFs of return intervals and the empirical PDFs calculated from normalized series of NYSE stocks (circles) and normalized series of currency exchanges (pluses). Al parameters of the model are the same as in previous figures, values of thresholds are as follows: $1.7, 2.0, 3.0, 4.0$. The straight lines are shown to guide the eye showing a power-law with exponent $3/2$.

We next analyze the daily returns data of 10 NYSE stocks obtained from Yahoo Finance, and also the USD historical exchange rates with currencies AU, NZ, POUND, CD, KRONER, YEN, KRONOR, and FRANCS traded on FOREX and obtained from the Federal Reserve. We first determine the appropriate scaling of the daily series of returns in the FOREX and NYSE exchanges. Because it is unlikely that those return series that exceed 50 years will be stationary, we normalize them by using a moving standard deviation procedure with a 5000-day time window. Each time series of all assets in both markets are normalized using this procedure. Figure 2(g) compares the normalized empirical PDFs with the model PDF, and Fig. 2(h) shows the PSDs. Notice that PSD of stock absolute returns in high frequency area has a slightly higher value than model and currency exchange PSDs. There is good agreement of PSDs in low frequency area.

Figure 5 shows that the unconditional PDFs of the daily scaled return intervals for NYSE stocks and FOREX exchange rates coincide for each threshold value. Note that in both NYSE and FOREX market the unconditional PDFs agree with the model PDFs. This indicates a high degree of scaling in the return intervals. The theoretical framework provided by our model

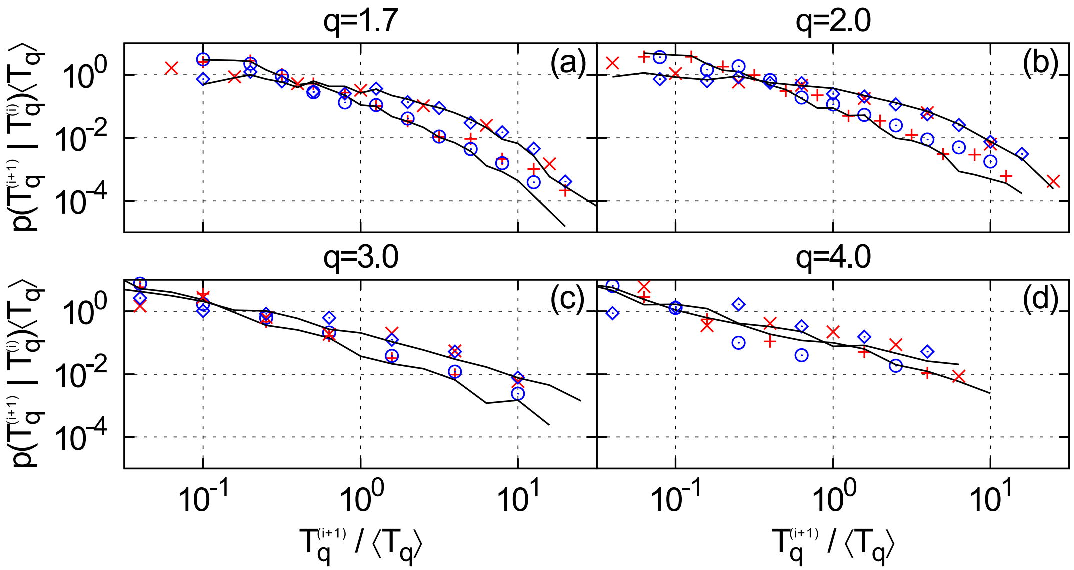

Figure 6. A comparison between the model (black lines) conditional PDFs of return intervals, $P(T_q^{i+1} \mid T_q^i)$ with $T_q^i \leq Q_1$ and $T_q^i \geq Q_8$ groups, and the corresponding empirical PDFs calculated from normalized series of NYSE stocks (circles and diamonds) and normalized series of currency exchanges (pluses and croses). All parameters of the model are the same as in previous figures, values of thresholds are as follows: $1.7, 2.0, 3.0, 4.0$.

is able to explain this scaling. Note that for the highest threshold value $q = 4$ the power-law exponent of the unconditional PDF in Fig. 5 deviates from $3/2$ and approaches 1.

Figure 6 shows that the conditional PDFs of the model agree with the conditional PDFs of the daily volatility return intervals records of both the NYSE and FOREX markets. When we increase the threshold, the conditional PDFs become closer and seems to overlap in both the empirical data and the model results, but we can not rule out that the seemingly overlap is due to the increased level of noise for high q.

Return intervals of monthly series for S&P500 index

We use data from an S&P500 monthly series spanning a 145-year period provided by Shiller[15] to demonstrate the behavior of return intervals for extremely long time-scales. Figure 7 shows that the above model, which reproduced the statistics of high frequency data, successfully mimics the PDFs of the volatility return intervals for even the longest time scales. We plot the empirical PDFs of return intervals for a nominal S&P500 and inflation adjusted series and compare them with the model series. The chosen threshold values range from 1.0 to 3.5 and represent several exponents of PDF. In the longest time-scales, the return interval distribution deviates from the $3/2$ power law for both the lowest and highest threshold values. Although the number of data points is limited, the model is able to capture these deviations and reproduce the behavior of index data.

Deviations from the 3/2 law

Because our model reproduces the statistical properties of empirical data for a wide range of assets and time-scales, we can use it to explain why increasing threshold q causes deviations from the theoretical 3/2 power law and the seemingly absence of memory in the conditional PDFs of return intervals. In particular, the model allows us to gradually switch off various noises and analyze how this changes the statistical properties of the return intervals.

The model conditional PDFs and the empirical data conditional PDFs overlap at approximately the same threshold values at which the unconditional PDFs deviate from the $3/2$ power law, for example, see Figure 5(d) and Figure 6(d). This phenomenon is stronger for larger time Δ scales, and we see no memory effects in the empirical S&P500 monthly series. Figure 7(d) shows the unconditional PDFs calculated numerically for several values of q, which resemble the exponential function discussed in Ref.[8] and obtained by reshuffling the absolute return time series. This indirectly confirms that the volatility return intervals for the S&P500 historical time series display no memory effect.

Our numerical simulations of the model suggest that the primary cause of the $3/2$ power-law behavior of the return intervals is the long-term SDE (see Eq. (6)). Other dynamic processes such as the speculative mood ξ (see Eq. (7)) and the intra-day seasonality contribute to the stability of this phenomenon. Although the exogenous noise in Eq. (1) causes the unconditional PDFs to deviate from the 3/2 power law, this noise is a necessary condition for the memory effect to

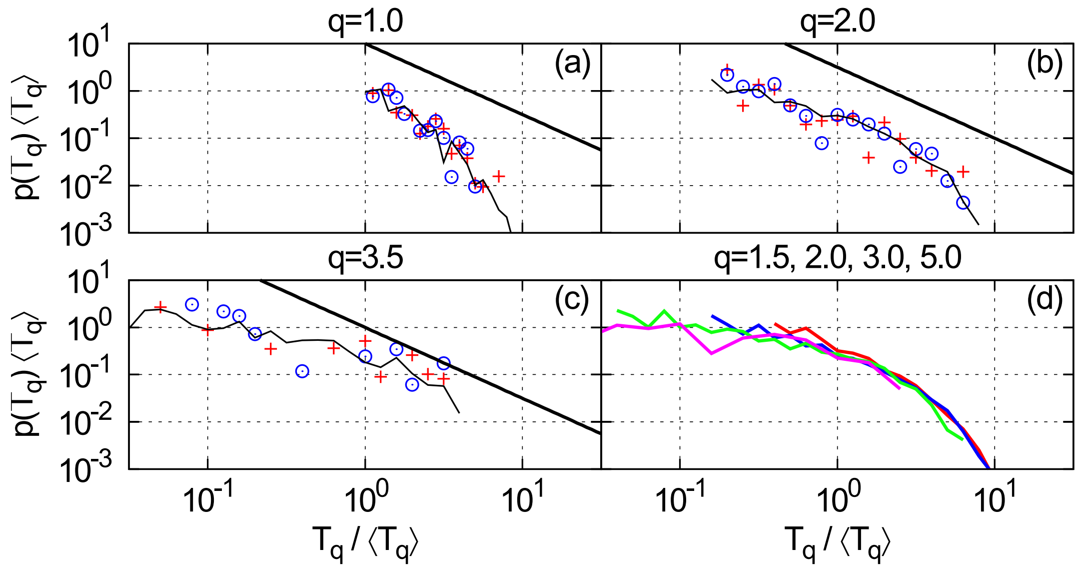

Figure 7. A comparison between the model (black lines) PDFs of return intervals and the empirical PDFs calculated from normalized series of historical $S\&P500$ data (circles) and the inflation adjusted series represented as real price (pluses). The straight lines are shown to guide the eye showing a power-law with exponent $3/2$. All parameters of the model are the same as in previous figures, values of thresholds q are as follows: (a) - 1, (b) - 2, (c) - 3.5. In subfigure (d) - numerical calculations of unconditional scalded PDFs for four values of threshold q: 1.5 (red), 2 (blue), 3 (green), 5 (purple).

emerge in conditional PDFs. From our numerical simulations we conclude that the deviations from the $3/2$ power-law and disappearance of the memory effect occur when the stochastic component is stronger than the dynamic component. This process of domination occurs when the threshold value is so high that the dynamic processes cannot reach it when the noise is switched off. Note that threshold q is measured in standard deviations of return, which grow approximately as $\Delta^{1/2}$. Dynamic processes quantified in σ_t of Eqs. (5) and (2) with this set of model parameters can only approach the threshold when its value is approximately equal to the standard deviation of the daily return time series. Thus when the thresholds are much higher the dynamic component is weaker than the stochastic component and the return intervals begin to deviate from the $3/2$ power law. The prevailing stochastic nature of the return time series destroys the memory effect, which requires that both dynamic and stochastic components be in the system.

Discussion

We have observed scaling and memory properties in the volatility return intervals in empirical data from the NYSE and the FOREX.[8–10] Our model is in agreement with the empirical return intervals that scale with the mean return interval $< T >$ as $P_q(T) =< T >^{-1} f(T/ < T >)$. The scaling function $f(x)$ is consistent with the power-law form $f(x) \sim x^{-3/2}$, which arises from the general theory of first-passage times in one-dimensional stochastic processes.[19,37] We recover the same scaling form for all assets analyzed from the NYSE and FOREX markets for return definition times Δ ranging from one minute to one month and for a wide range of thresholds q, which represent the power-law component of the empirical return series. Our model also captures the deviations of the volatility return interval PDF exponent from the main value $3/2$ and explains the origin of these deviations.

We also have observed that at low q values at the beginning of the power-law component of the empirical return series, for both one-minute and one-day periods, the conditional PDFs $P(T_q^{i+1} \mid T_q^i)$ for $T_q^i \leq Q_1$ and $T_q^i \geq Q_8$ are different, and this indicates the presence of a memory effect. Our model suggests that this effect is caused by a complex interplay of all the noises included in the system. The necessary condition for the memory effect is the presence of long-term agent dynamics and exogenous noise. High threshold q values seem to cause the memory effect to disappear as the stochastic component of the volatility begins to prevail.

When we compare the results of our model with the monthly data from the S&P500 we are able to extend our research on the scaling properties of the volatility return interval up to the natural limits of the phenomenon. We thus suggest that the deviations of the PDF exponent from $3/2$ are caused by an interplay between agent dynamics and exogenous noise. The standard deviations of return in the S&P500 monthly series are so high that the dynamic component of the system becomes negligible and the stochastic componen dominates. This causes the exponential scaling functions of the return intervals and the disappearance of the memory effects.

We have found that the statistical and scaling properties such as the observable power-law behavior in the returns can be explained using non-linear stochastic modeling.[7] The extreme power-law scaling properties observed in all assets, markets, and time-scales can be explained by the scaling properties of a class of nonlinear stochastic differential equations described in detail in Refs.[16,33] Our model here is based on the herding interactions of agents, and its macroscopic version is derived as a system of stochastic equations. These equations might be the origin for the power-law properties of the power spectral density and signal autocorrelation represented by long-range memory.

We have also demonstrated that the model can be scaled for markets with trading hours of different duration and that the duration can be extended to a 24-hour day. This allows a general approach to empirical data scaling and the retrieval of the same power law properties in different markets and different assets.

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Acknowledgements

This work was partially supported by Baltic-American Freedom Foundation and CIEE.

Author contributions statement

VG and AK performed analyses, VG, SH, AK, BP and HES discussed the results, and contributed to the text of the manuscript, SH, BP and HES reviewed and approved the manuscript.

Additional information

The authors declare no competing financial interests.

The corresponding author is responsible for submitting a Competing financial interests on behalf of all authors of the paper. This statement must be included in the submitted article file.