Towards Exact Nonextensive Solutions of the American Style Options 4: Infinitesimal Linear Evolution of the Early Exercise Premium with Respect to the American Derivative Can Lead to Black-Scholes Like Closed Form Solutions.

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Abstract:

Towards obtaining an understanding of the additional non-market source of risk, the random boundaries and slack or compensating functions due to secondary sources of early exercise risk, and inequalities inherent (therefore) to portfolios and PDEs of the American style derivative (options), we have recently submitted 3 different approaches [5] and letters discussing details and deriving possible solutions. In this letter we present a fourth discussion and result(s). We specifically focus on the early exercise (risk) premium, this a monetary measure of the early exercise source of secondary non-market uncertainty, and utilize this idea of monetary valuation of non-market early exercise uncertainty (risk) to overcome the inequality portfolio and inequality PDE problem. We do this by relating the equality portfolio of the European style derivative to the American style directly via the A-E=p relation between American and European derivatives. We furthermore under an assumption of linear variation of early exercise premium with respect to American option derive an American Black-Scholes like PDE model and obtain a closed form solution for the same which parallels the traditional Black-Scholes formula.

Discussion:

In this letter we specifically focus on the early exercise (risk) premium p(x,t), and connect this idea to the preceding discussions and letters. We utilize this definition of a premium, the difference between the American and European derivatives A-E=p, to approach American valuation from the European style derivative theory and methods, these all equality relations! In that this is different from letters I-III in that in those approaches, all derivations were from the American derivatives relations these inequality relations! The difficulties then were those of somehow transforming those inequalities to equalities. Again…here we begin from European relations these equalities and retain that feature throughout to a closed form solution.

We furthermore under a strict assumption of linear evolution of early exercise premium with respect to American option, we derive an American Black-Scholes like model and obtain a closed form 'exact' solution for the same.

The early exercise premium is assumed to arise as

$$A(x,t) - E(x,t) = p(x,t) \qquad (1.a.)$$

With A the American option, E the European, and the p the premium for the possibility of early exercise. We generally write this as a function of (x,t) with x the stock price say, and specialize to time or price dependence (only) in the following.

And it must be said that most if not all recent approaches to describing American style options, including our recent 3 letters and descriptions, are some sort of variation and or deviation away from the known European style pricing. So this statement can also be written down, as

$$dA = E\,|_{A|_{p_o}} + \frac{\partial A}{\partial E} dE \quad . \tag{1.b.}$$

The relation between the American style and the European style to first order (higher, second order and beyond corrections are simply expanded) is obtained from expanding in premium and in European as

$$\begin{aligned} dA &= A\,|_{p_o} + \frac{\partial A}{\partial p} dp \\ dA &= A\,|_{E_o} + \frac{\partial A}{\partial E} dE \\ dE &= \frac{\partial E}{\partial p} dp \quad . \\ dp &= \frac{\partial p}{\partial E} dE \\ A\,|_{p_o} &= E \end{aligned} \tag{1.c.}$$

The relations are just statements about the A-E=p and that the 'initial' premium which could be zero is where the A=E derivatives coincide and which we focus on as being the t=0 initial contract issuance time however this can be relaxed for other approaches. The linear relationship can be cast as a variation (see below) and can be made nonlinear (not shown). We have not yet defined how these A, p evolve and in relation with E. We do this in the subsequent derivation for an American style Black-Scholes like model below.

Now we wish to derive a Black-Scholes model for an American style option, Note that we do not delve into the optimal exercise time/price and boundary issue(s) and therefore the differences between modeling the American call or put from that perspective. We assume that the underlying stochastics of say the index or stocks obtain market uncertainty or risk, the risk is Legendre transformed (hedged) by the portfolio relations as in the European (B-S) portfolio (equality), however the additional uncertainties due to risk bias and early exercise perfection of knowledge regarding rational early exercise introducing a second source of risk and uncertainty therefore rendering the usual portfolio relations into an inequality. To compare American and European portfolios and hedges:

$$\begin{aligned} \pi(t) &\geq A - \delta x \\ \Pi(t) &= E - \Delta x \end{aligned} \quad . \tag{2}$$

We recently supposed we can add an additional uncertainty instrument such that the inequality can be made into an equality [5], We discussed this at length from 3 different perspectives infact, stating that the uncertainty can be viewed as i) a random boundary condition as in letter I ii) a surplus or slack function as in a generalized linear programming type of approach to functional inequalities as in letter II and iii) as a Bogoliubov inequality like relationship and with the introduction of the 'thermodynamics of finance' as in letter III. We additionally peppered these letters with appendices attempting to add additional though lesser views and connections. Other generalizations of European style Black-Scholes to more accurate underlying stochastics and additionally to early exercise sources of uncertainty modeling are also applicable as we referred to these in some detail in I-III and these are in [1-9]. The nonextensive statistics especially have proven useful both in types of secondary early exercise uncertainty modeling and in generalizations of the derivative market uncertainty sources terms in the PDEs and towards accurate and market replicating derivatives pricing modeling and mathematical descriptions.

In all of these cases, we sought to bridge the gap from inequality to equality portfolio relating American to European style portfolios and then from this deriving PDEs of derivatives' evolution, seeking to provide justification from the microscopic i.e. stochastic to the macroscopic PDEs, PDFs and thermodynamical state functions levels.

These several letters have however inspired a possibly simpler approach, the fourth approach that of this letter.

In this simpler approach, we resort to the definition of the early exercise premium A-E=p as the difference in monetary value of the American and European derivative, and then make certain assumptions regarding this premium's evolution (variation). That is, we take this variation to first order in the following for simplicity however it is clear higher order terms can straight forwardly be included.

This approach of relating A=E+p and then how p evolves with respect to w.r.t. A allows us to bridge the gap from equality to inequality portfolios or from European to American derivatives. We detail this in the following.

The European derivative portfolio equality relation is as from Eq.(2) and we rewrite it with the aid of the premium relation between American and European derivatives as

$$\begin{aligned} \Pi(t) &= E - \Delta x \\ \Pi(t) &= A - p - \Delta x \end{aligned} \quad . \tag{3}$$

And we vary this relation noting that the portfolio is time dependent (only) and evolves at the risk free rate

$$\begin{aligned} d\Pi &= r\Pi dt \\ d\Pi &= dA - dp - \Delta dx \end{aligned} \, ' \tag{4}$$

And we vary the dA as usual to 2nd order in price and first in time, however we vary the premium w.r.t. with respect to the derivative to first order, this a change of variables equivalently as

$$
\begin{aligned}
d\Pi &= (1-\frac{\partial p}{\partial A})dA - \Delta dx \\
dA &= \frac{\partial A}{\partial t}dt + \frac{\partial A}{\partial x}dx + \frac{1}{2}\frac{\partial^2 A}{\partial x^2}dx^2 \\
\Delta &= (1-\frac{\partial p}{\partial A})\frac{\partial A}{\partial x} = \frac{\partial E}{\partial x}
\end{aligned}
\quad . \qquad (5)
$$

Eliminating the dx variations, equivalently the 'Delta Hedge' which as shown is the European delta, and then dividing out the (…) term in the parenthesis relating premium to American option,

$$
\begin{aligned}
&(1-\frac{\partial p}{\partial A})^{-1} r(A - p - x\frac{\partial A}{\partial x}) = \frac{\partial A}{\partial t} + \frac{b^2(x,t)}{2}\frac{\partial^2 A}{\partial x^2} \\
&(1-\frac{\partial p}{\partial A})^{-1} = (\frac{\partial A}{\partial E}) \\
&r[(\frac{\partial \ln E}{\partial x})^{-1} - x]\frac{\partial A}{\partial x} = \frac{\partial A}{\partial t} + \frac{b^2(x,t)}{2}\frac{\partial^2 A}{\partial x^2} \quad . \\
&\frac{\partial A}{\partial t} = r[(\frac{\partial \ln E}{\partial x})^{-1} - x]\frac{\partial A}{\partial x} - \frac{b^2(x,t)}{2}\frac{\partial^2 A}{\partial x^2} \\
&\frac{\partial A}{\partial t} = ra(x,t)\frac{\partial A}{\partial x} - \frac{b^2(x,t)}{2}\frac{\partial^2 A}{\partial x^2} \\
&(\frac{\partial \ln E}{\partial x})^{-1} = \frac{E}{\Delta_E}
\end{aligned}
\qquad (6)
$$

Note that we transformed partial differentiation variables in the above such that the differentials were according to the price x. The derivation under the assumption that the variation of the premium w.r.t. the option is linear allows us to gather terms and rewrite the European portfolio equality in terms of the American derivative. The result is a modified backwards Fokker-Planck PDE that is a Black-Scholes equation for the American style derivative. While traditional Black-Scholes with a log normal process obtains the diffusion and drift coefficients as

$$
\begin{aligned}
&b^2(x,t) = D_o x^2 \\
&a_E(x,t) = r(E - x\frac{\partial E}{\partial x}) = r(\frac{E}{\Delta_E} - x)\frac{\partial E}{\partial x} \\
&a_A(x,t) = r((\frac{\partial \ln E}{\partial x})^{-1} - x)\frac{\partial A}{\partial x} = r(\frac{E}{\Delta_E} - x)\frac{\partial A}{\partial x} \\
&a_A - a_E = a_p \qquad , \qquad (7) \\
&\frac{\partial (A-E)}{\partial x} = \frac{\partial p}{\partial x} = \frac{a_p}{r(\frac{E}{\Delta_E} - x)} \\
&a_p \propto p \\
&p \propto e^{\int_{x_o}^{x} \frac{1}{r(\frac{E}{\Delta_E} - x")} dx"}
\end{aligned}
$$

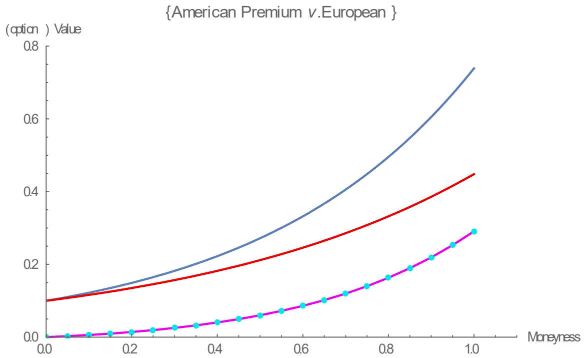
Returning to the PDEs, usually we absorb the constant drift portion rE into the solution (E(x,t)->g(x,t;r)) and then work on solving the PDE for g. Here we showed the 'unabsorbed' drift portion to compare A and E drifts. With the nonlinear drift in lieu of a constant drift 'r' this is no longer possible. The American style derivative introduces therefore an additional nonlinear term to the drift coefficient. Thus the effects of an early exercise premium, which in previous letters were noticeable as additional diffusion coefficients only or interpreted as additional sources of uncertainty due to early exercise, here are introduced as additional drift terms or effects. We remind the reader that drift+diffusion are partition-able usually in stochastic SDEs differential equations via transformations by Ito's formula as dx=a(x,t)dt+b(x,t)dW(t) can be rewritten as dx'=B(x',t)dW(t) or dx''=c(x'',t)dt+dW(t) an so on, and at the PDE level by PDF and Jacobian transformations from one form to another say known form as P(x,t,..)dxdt..=G(y,T,..)dydT.. with say G known.

This then is a drift only effect result whereas previously we introduced diffusion only effects results. However we stress that both approaches stem from modeling the early exercise additional freedom (and additional sources of uncertainty, i.e. risk).

And the reader should notice that the two PDEs share a common factor in-drift, and as we chose to write it in terms of the European delta hedge to point this out. In the case of European PDEs this is as we mentioned absorbed into a pre exponential. Not so for the American PDE. And as the two share a common factor and can be related to premium, we also derived a proportionality expression for the premium which at this point is merely proportional yet which we feel is indicative of how the premium evolves in trend and here as expressed by what it evolves by that is the European value and European Delta hedge. Other expressions for the premium are obtainable.

Let us segue briefly away from the main thrust of this letter the derivation and discussion of the American style derivative solutions (if any), and discuss further the import of the A-E=p premium.

This premium evolves as an exponentially increasing value, with increasing moneyness. This has been reported elsewhere, the shape of this is



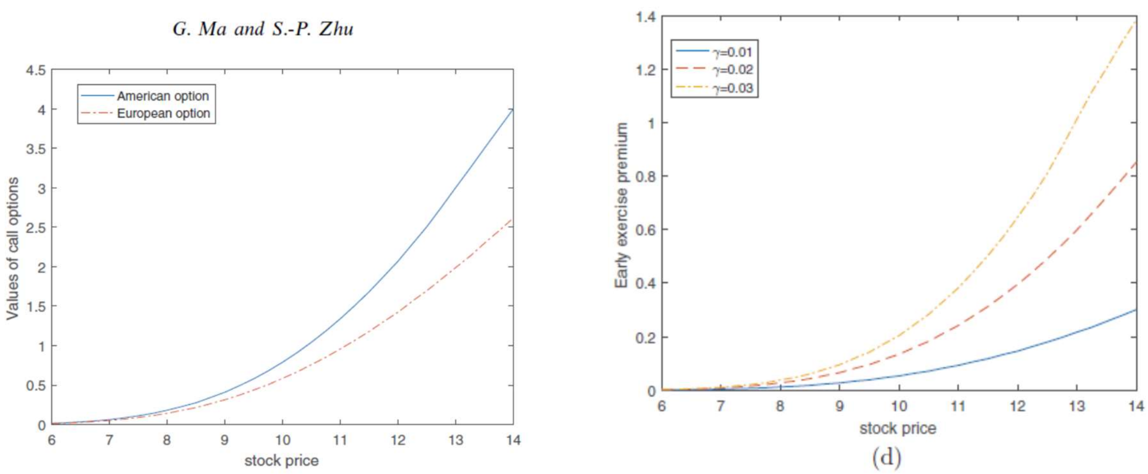


Fig.1. Premium for A and E as a function of moneyness, in arbitrary units, for illustration, A(Blue) E(Red) and p(purple). The second figure is a Ma and Zhu calculation, for reference. From J. of Appl. Mathematics (2018) vol.29, pp.494-514, reproduced without permission yet cited.

The American style Black-Sholes B-S PDE and a solution of the Call or put can be obtained by two well-known methods which we will pursue here. First the Cox method integral solution

$$A_C(x,t) = \int \theta(K - x')A(x,t \mid x't')dx' \quad , \tag{8}$$

Where the step function (Heaviside) selects the strike price value boundary of moneyness for a 'Call' option this max{x-K,0}. A put option can be solved for similarly with the appropriate boundaries from the (again..) appropriate PDE.

The second and more familiar method is the transformations method. The Black-Scholes theory's pricing formula or PDE is transformed to the standard diffusion or heat equation that is utilized in obtaining exact solutions to the B-S equation. The usual or traditional solution is [3]

$$
\begin{aligned}
C &= x(t)N(d_1) - Ke^{-rt}N(d_2) \\
d_1 &= \frac{\ln\frac{x}{K} + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \\
d_2 &= d_1 - \sigma\sqrt{t} \\
d_2 &= \frac{\ln\frac{x}{K} + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}
\end{aligned} \quad . \qquad (9)
$$

The Call option solution is comprised of cumulative distribution functions N(..). The variables for 'd' are the transformations towards a diffusion PDE as we discuss below.

The second method depends on change of variables, for example to a diffusion equation

$$
\begin{aligned}
\tau &= T - t \\
u &= Ce^{r\tau} \\
y &= \ln(\frac{x}{K}) + (r - \frac{1}{2}\sigma^2)\tau \, . \\
\frac{\partial u}{\partial \tau} &= \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial y^2}
\end{aligned} \qquad (10)
$$

The solutions for this equation taking into account boundary conditions:

$$
\begin{aligned}
u(y,\tau) &= \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} u_o(y')e^{-\frac{(y-y')^2}{2\sigma^2\tau}} dy' \\
C(x_T, T) &= \max\{x_T - K, 0\} \\
u(y,0) &= u_o(y) := K(e^{\max\{y,0\}} - 1) = K(e^y - 1)H(y)
\end{aligned} \quad . \qquad (11)
$$

It will be instructive to apply the similar to our nonlinear drift coefficient Black-Scholes (American) equation. Therefore we proceed in several steps to transform the PDE backwards Fokker-Planck equation Eq.(6) to a diffusion equation similar to Eqs.(10-11).

First we transform to a new coordinate system that results in a diffusion only PDE (heat equation). We state at the outset here that this will not allow a closed form solution as in B-S , this due to the difficulty of re boundarization of the transformed variables to the terminal time and strike. However we present this in order to illustrate this difficulty, and to point out where alternative numerical schemes can be applied. The reader interested in Black-Scholes like closed form solutions (Eq.(16) below) by yet another method of transformations (mixed temporal-price) should skip to that section below.

The variable (price) transformations are:

$$
\begin{aligned}
& ra(x,t)\frac{\partial y}{\partial x}\frac{\partial A}{\partial y}=\frac{\partial A}{\partial y} \\
& \frac{\partial A}{\partial t}=\frac{\partial A}{\partial y}-\frac{b^2(x,t)}{2}(\frac{\partial y}{\partial x})^2\frac{\partial^2 A}{\partial y^2} \\
& \frac{\partial A}{\partial t}=\frac{\partial A}{\partial y}-\frac{b^2}{2r^2a^2}\frac{\partial^2 A}{\partial y^2} \\
& \frac{\partial}{\partial t'}A(y,t\,|\,y',t')=-\frac{\partial A}{\partial y'}+\frac{1}{2}\frac{\partial^2 \frac{b^2}{r^2a^2}A}{\partial y'^2}
\end{aligned}
\qquad (12)
$$

And where the coordinates are related by transformation as

$$
\begin{aligned}
& y(x,t)=\int_0^x \frac{1}{ra(x'',t)}dx'' \\
& a(x,t)=(\frac{E}{\Delta_E}-x)
\end{aligned}
\qquad (13)
$$

And then where we in Eq.(12) utilized the forward-backward solution symmetry of the two-point solution. From the forward Fokker-Planck PDE we formally relate to the microscopic SDE stochastic differential equation (Ito sense) level which we use to de-drift and then the resulting non-constant diffusion coefficient diffusion PDE is further transformed to a constant diffusion coefficient diffusion equation as in the B-S Black Scholes case…this three step (or more, depending on view) transformation then arrives at the diffusion or heat equation which is solved by the following:

$$
\begin{aligned}
& dy'=1dt'+\frac{b}{ra}dW(t') \\
& z'=(y'-t') \\
& dz'=\frac{b}{ra}dW(t') \\
& \frac{\partial}{\partial t'}A(z,t\,|\,z',t')=\frac{1}{2}\frac{\partial^2 \frac{b^2}{r^2a^2}A}{\partial z'^2}
\end{aligned}
\qquad (14.a.)
$$

The last PDE is a diffusion equation, which we arrived at by the transformation of its backwards (forward not shown…one must take care between these forward and backwards forms and transformations, noting that we are now working with the forward PDE) form

$$\frac{b^2}{r^2a^2}(\frac{\partial\lambda}{\partial z})^2\frac{1}{2}\frac{\partial^2 A}{\partial\lambda^2}=\frac{1}{2}\frac{\partial^2 A}{\partial\lambda^2}$$
$$\lambda(z,t)=\int^{z}\frac{ra(z'',t)}{b(z'',t)}dz'' \quad . \qquad (14.b.)$$

Its solution is a Gaussian as

$$A(\lambda,t\,|\,\lambda',t')=\frac{e^{-\frac{(\lambda-\lambda')^2}{2(t-t')}}}{\sqrt{4\pi(t-t')}} \quad . \qquad (15)$$

It remains therefore to solve for the say Call or Put options that is given the boundaries maxima of in-the-money.

However unlike the simple B-S model the transformations are not logarithmic, and are not clearly amenable to re-boundarization (Eq.11.). We in fact derived this approach in order to show that difficulty and also to point the way forward as the problem is made clear. Additionally we suggest that a calculator would (probably) be in order at this point. Not to worry though, even the log-normal process closed form or exact solution of Black-Scholes which we will next derive buy mixed transforms will/requires a calculator (at least, maybe some software packages) as the author is not aware of any human being who can calculate a cumulant (the N(..) in Eq.(9)) with sufficient monetary trading accuracy in their mind or on the back of an envelope.

An alternative is to calculate the PDEs directly numerically, say by a discretization scheme or FE scheme. This is in fact always a fruitful course in nonlinear PDEs and we pursue this in future works' sections on numerical results.

And another alternative and one which will yield results of closed form Black-Scholes like solutions is that of mixed time-like transforms. Consider that the American PDE is the European PDE plus a second and non-constant drift coefficient corresponding to the premium's effects in the backwards Fokker-Planck frame. We can therefore try to absorb the nonlinear (non-constant) drift component into a variable or parameter such as the time parameterization, or temporal evolution. This leaving the European components, here a Black-Scholes log-normal process based drift and diffusion coefficients containing PDE (for simplicity…other and more accurate diffusion processes are published elsewhere for example nonextensive statistics distributed sources [6-9]), as is or as they are and this overall leading to a mapping to a European (overall) style PDE as follows:

$$
\begin{aligned}
&\frac{\partial A}{\partial t} = r\frac{\partial A}{\partial E}(A - p - x\frac{\partial E}{\partial x}) - \frac{b^2(x,t)}{2}\frac{\partial^2 A}{\partial x^2} \\
&\frac{\partial A}{\partial t} = r\frac{\partial A}{\partial \ln E} - rx\frac{\partial A}{\partial x} - \frac{b^2}{2}\frac{\partial^2 A}{\partial x^2} \\
&r\frac{\partial A}{\partial \ln E} = r\frac{\partial A}{\partial t}\frac{\partial t}{\partial \ln E} \\
&(1 - r\frac{\partial t}{\partial \ln E})\frac{\partial A}{\partial t} = L_{g_E}A \\
&(1 - r\frac{\partial t}{\partial \ln E})\frac{\partial \tau}{\partial t}\frac{\partial A}{\partial \tau} = \frac{\partial A}{\partial \tau} = L_{g_E}A \\
&\tau(x,t) = \int_0^t \frac{1}{(1 - r\frac{\partial t''}{\partial \ln E(x,t'')})}dt'' \\
&\frac{\partial A}{\partial \tau} = -rx\frac{\partial A}{\partial x} - \frac{D_o x^2}{2}\frac{\partial^2 A}{\partial x^2} \\
&D_o = \sigma^2 \\
&\frac{\partial}{\partial \tau}A(\lambda, \tau \mid \lambda', \tau') = \frac{\sigma^2}{2}\frac{\partial^2 A}{\partial \lambda^2} \\
&x = e^{\lambda} \\
&A(x,t) = \max\{x - K, 0\} \\
&A(\lambda, \tau) = K\max\{e^{\lambda} - 1, 0\} = K(e^{\lambda} - 1)H(\lambda) \\
&A(\lambda, t) = Ke^{\lambda + \frac{1}{2}\sigma^2\tau}N(d_1) - KN(d_2) \\
&d_{1/2} = \frac{1}{\sigma\sqrt{\tau}}[(\lambda + \frac{1}{2}\sigma^2\tau) \pm \frac{1}{2}\sigma^2\tau]
\end{aligned}
. \qquad (16)
$$

Therefore in Eq.(16) and with the traditional Black-Scholes formula [3] and with the mixed transformation in time, a mapping to Black-Scholes' closed form solution is obtained. We in fact can claim that a solution in closed form generalizing the traditional formula is obtained for the American style option. The differences between the American and European derivative (option) from this approach are clear. The European has monotonic temporal parameterized evolution t while the American has a mixed temporal nonlinear parametric function evolution 'tau' τ(x,t) which incorporates the effects of the early exercise premium.

Conclusion:

In this letter we approach again the problem of the American style derivative (option) pricing from yet another perspective that of the price premium due to early exercise possibility.

Avoiding the issues of a potential optimal exercise boundary and issues of put and call equivalence to European options if dividends are paid etc., we merely assert that the American style option contains two (at least) sources of uncertainty…that of market underlying uncertainty and secondly that of early exercise or random boundary uncertainty. This latter producing a price premium, due to the early exercise possibility and degree of freedom.

The risk premium relates the American and European options as A-E=p in some form. Others have produced formula for the early exercise premium yet these approached from other perspectives. Here we obtain additional forms for the premium, yet as an aside as our interest is the utilization of this formula A-E=p to obtain an expression for the American style option.

That is, starting from a European portfolio $\Pi = E - \Delta x$, we substitute the E=A-p expression into the equality portfolio and therefore we immediately have an equality American portfolio relation $\Pi = A - p - \Delta x$ (obviously) albeit with unknown p.

We shift to a purely American equality portfolio by doing away with the price premium in favor of known European quantities. We do this by a linear approximation here though other and higher order approximations are possible, and ones we may return to in future work in order to obtain closer convergence with market data.

However even the linear first order expansion of the premium in terms of the American option as in $dp = \frac{\partial p}{\partial A} dA$ in the European style equality portfolio equation variation allows us to hedge by European means, while then incorporating the premium's effects into the drift coefficient of an equality PDE for the American style option. The disadvantage now being that the drift becomes nonlinear in that component (it can be cast as a potential, source/sink, or as a drift coefficient).

However now this is mathematics not economics of finance. The nonlinear terms can be absorbed into the temporal parameter evolution or variable(s) by suitable transformations, we show two such transformations and discuss other solution methods those of the integral and PDE numerical approaches, and point out the difficulties inherent in them as well as find a pathway towards obtaining a closed form solution such as the traditional log normal process Black-Scholes, however whereas the evolution in monotonically constant in time in B-S, this is now a nonlinear function of temporal-price mixed evolution.

We remind the reader that we have sought the effects of the early exercise, here included by a premium as obviously the additional early exercise degree of freedom changes the American to European price relationship making the American style more expensive (more control of risk, higher price), by other means such as additional diffusion components in the European PDE, and by slack and compensation function that transform inequality relations to equality relations. Therefore our current approach of premiums as drift, will have to be compared to and related to our previous approaches. Here we started from European equality portfolios, in previous letters we started from American style inequality portfolios, in future work we intend to connect these further and show more precise relations between say slack or compensation function and premium drift coefficient functions.

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And see letters I-III http://www3.unifr.ch/econophysics/?q=content/towards-exact-nonextensive-solutions-american-style-options-ii and http://www3.unifr.ch/econophysics/?q=content/towards-exact-solutions-american-style-options on the unifr.ch/econophysics web pages.
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