Towards Exact Nonextensive Solutions Of The American Style Options II

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Abstract:
Progress towards an exact solution(s) of the American style call or put option on the same footing as the European (Vanilla) call or put option has been reported by us recently, this under certain random (exercise) boundary assumptions . In this letter we present several (more) approaches towards deriving such exact solutions, functions that are obtained from & which are based on applying boundary conditions that are here additionally random in time. We derive under certain sets of limiting assumptions several closed form extensive & nonextensive statistics solution(s).

Introduction:

We have recently reported the mathematical derivation of one set of possible exact solution(s) of the American style option that utilizes the assumption that the early possibility of exercise of the American style option can be equated to random in time boundary conditions [1]. Secondly that the American style option is from portfolio arguments & hedging & removal of drift an inequality PDE of the Black-Scholes type. In equation form [1] this is written as

$$\frac{\partial}{\partial t} f(x,t) - r(1 - x\frac{\partial}{\partial x}) f(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} f \le 0 \qquad (1)$$

and where at 0 it coincides with the European style option formula, and where the usual portfolio construction of a European style call option is used to derive the LHS Black-Scholes style PDE yet this is no longer equated with zero [1,2].

Thus the American style option in Eq.(1) can schematically be represented as evolving from t=0 to t=E exercise E<T intermediate & random -like time [1-3]:

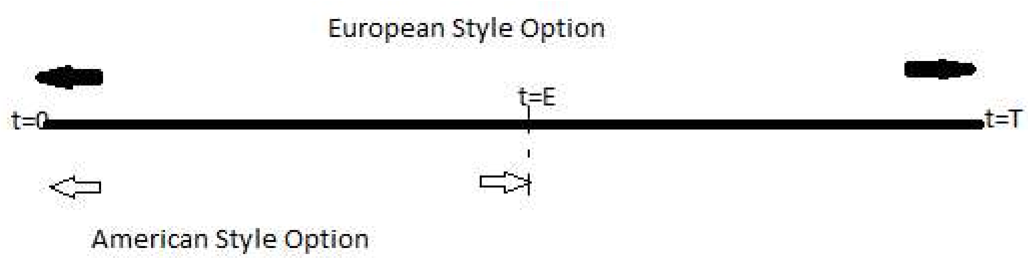

Fig.1. Schematic European & American options evolution. As E->T the American style option unexercised resembles more & more the European option until at E=T they are identical.

Additionally In the previous letter the random-in-time boundaries (E in Fig.1.) of possible exercise forced a transformation of the PDE to a time-diffusion PDE equation. This is useful in deriving the fundamentals of why & how random boundaries enter into the mathematical structure. However this also prompts new insights into simpler & direct derivations. In this letter we will derive these simpler results as informed by [1].

Introduction II:

This section is nearly identical to the section II intro as in [1] and is merely included for a self contained presentation. We omit the trees binomial & trinomial models discussion found in [1] but note that a tree model can be looked at as an evolution (t=t1,t2,t3,...,tn=T) by the European style option formula from t1 to t2 , and where at t2 an exercise decision is made (i.e. t2=E) at which point if unexercised another European style option formula evolution is made to t3 (now t3=E) and so on.

....Also we want to point out that the Eq.(1) for the American style option is an inequality, but the equation can be made into an equality with zero as

$$\frac{\partial}{\partial t} f(x,t) - r(1 - x\frac{\partial}{\partial x}) f(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} f + L(x,t) = 0$$
$$L \geq 0 \qquad (4)$$

where the 'deficit function' L(x,t), is a function or a differential operator acting on a function, in detailed view, which changes the form of the inequality identically such that an equality is satisfied, and which is greater than or equal to zero itself, and has the interpretation that it is the uncertainty associated with the early exercise &/or the random boundary conditions, &/or the free boundary problem these several overlapping points of view, & this generally also the view both from the trinomial (or binomial) 'trees' discrete process which was our 'conceptual' starting point in [1]. This equality is made by the composition as by superposition of operators &/or functions with the inequality portion (the otherwise European style option formula portion), which portion is itself made up of/with the superposition of first order operator(s) & function(s) of drift & whereas the second order operator(s) & function(s) of diffusion (or uncertainty & the 2nd moment of f) & in general by extension of the process of superposition can be made at the function(al) or operator levels. That is, by information theoretic arguments, an information measure can be composed that adds observables (as operators &/or functions as

$$D[< I(t) >] + D[\beta(< \vartheta_1 > + ... + < \vartheta_n >)] == 0 \qquad (4a)$$

where in the extremization in Eq.(4a) <I(t)> is the statistically <..> averaged (integrated &/or summed) information or equivalently negentropy, & the observables posited as sufficient to capture the dynamics of the system range as desired from 1->n & are composed in superposition with Lagrange multiplier weights for proportionality. In this case at least observables 1 & 2 are the 1st & 2nd moments corresponding to the drift coefficient & diffusion coefficient of the Forward Fokker-Planck F-P PDE corresponding to the backwards F-P PDE of Eq.(1) (with constant r potential absorbed), & additionally therefore L in Eq.(4) corresponds to an observable in Eq.(4a) which then obtains a Forward form which when the backwards F-P is derived is L. A second supporting argument is that of superposition of operators/functions in Hamiltonian theory which Eq.(4) can be viewed as a parameterized (in t) form of.

And it should be noted that equation Eq.(4) is onto itself a starting point for tries for derivation of an American Black-Scholes equation, as the Eq.(1) obstacle to a solution is the inequality (otherwise a solution is obtained as for the European Black-Scholes formula), & 'forcing' an equality as by analogy to NLP nonlinear programming whereas deficit & surplus functions are utilized to transform inequalities to equalities is being applied here, from one view point. Another point of view we have addressed in [1] & maintain here is that the portfolio from which Eq.(1) is derived is now augmented with an asset accounting for the hedging of the risk due to early exercise (or of holding the option further, that of not early exercise) which corresponds to L identically.

As an example one can require that a) L(x,t) is a function b) L(,t) is an operator(s) on a function(s) & up to 2nd order (see discussion following).

And for completeness at least of the economics that motivate this mathematics, what does the L-function correspond to in terms of real hard assets & equities & hedges & riskless investments...meaning that given a portfolio

$$\Pi(t) = \Pi_o e^{rt} \leq f - x\Delta$$

$$d\Pi \leq df - dx\Delta$$

$$d\Pi \overset{?}{=} df - dx\Delta + F[L(x,t)] \quad \text{(p)}$$

requiring that at all times the equality is re-instated as by a hedge (the delta in the above portfolio) or financial instrument or asset or equity is very much a question of economics and must be addressed as such. An alternative form of the function is that of a functional, a function that is a self-referring (deficit or surplus) modifier(s) and being the L equalizing function (i.e.,functionals of f F(f)), but this too has an economic interpretation...one of discounts or premiums paid out or taken in. We return to these questions in future work. We introduce a third interpretation in the following discussions & derivations that is in accord with our (& others') previous work on generalized nonextensive Black-Scholes derivatives pricing formulas, that of a nonextensive apriori nonlinear compositions in Eq.(4a) which results in Eqs.(p) such that (European, but as discussed below, also American) an equality portfolio is achieved.

Model I, Derivation:

We have therefore discussed the current letter's inherent assumptions in the introduction, where we obtain an L-function equality modified B-S Black-Scholes PDE for an American style option. This option pricing formula is nearly identical to the European or vanilla style option formula and as E->T the two become identical as L->0, a useful limiting concept to consider & one which we will take into consideration later.

However it has the deficit function L as a modification. This function L can be viewed in detail better if we first recast Eq.(4) as a forward standard form Fokker-Planck PDE 2nd order partial differential equation as follows:

$$\frac{\partial}{\partial t'} g(x',t') = -\frac{\partial}{\partial x'}[rx'g] + \frac{1}{2}\frac{\partial^2}{\partial x'^2} g + \lambda(x',t')$$
$$f(x,t;x',t') = e^{r|t-t'|} g(x,t;x',t') \quad (5)$$

Here we utilize the fact that the two-point function solves both the forward & backward PDEs, & remove or absorb the constant 'potential' r into the solution, & redefine the transformed equality function L as Lambda, to write the (European style+Lambda=American Style, equality) F-P PDE which will enter as the 'unperturbed' operator in superposition with the 'perturbation' Lambda in the following.

1) Lambda is a potential-like perturbation. In this case we write for the one-point g(x',t') forward PDE obtained from g(x,t;x',t') two-point function & PDE

$$\frac{\partial}{\partial t'}g(x',t')=-\frac{\partial}{\partial x'}[rx'g]+\frac{1}{2}\frac{\partial^2}{\partial x'^2}g+\tilde{\phi}(x',t')g(x',t')$$
$$\lambda(x',t')=\tilde{\phi}(x',t')g(x',t')\geq 0 \qquad \text{(5a)},$$

and we can resort to several methods of solution described in the following. For example Eq.(5a) is

solved by well-known Green's functions & perturbation methods (the two-point Eqs.(5 & 5a) acquire delta function sources $\delta(x-x')\delta(t-t')$ on their RHS)

$$g(x,t;x',t')=g_o(x,t;x',t')+\int g_o(x,t;x'',t'')\tilde{\phi}(x'',t'')g(x'',t'';x',t')dx''dt' \qquad \text{(5a.1)}$$

where g_o is the 'unperturbed' by the potential forward Green's function which solves

$$[\frac{\partial}{\partial t'}+\frac{\partial}{\partial x'}[rx']-\frac{1}{2}\frac{\partial^2}{\partial x'^2}]g_o(x,t\,|\,x',t')=\delta(x-x')\delta(t-t') \qquad \text{(5a.2)}$$

and we use the traditional notation for the g_o two-point sourced transition probability which is parameterized by t,t'.

We shall return to these methods in future work.

2) Lambda is a first order drift like operator acting on a function or functions. In this case we write

$$[\frac{\partial}{\partial t'}+\frac{\partial}{\partial x'}(rx')+\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial x}(\theta(x',t'))]g(x,t\,|\,x',t')=\delta(x-x')\delta(t-t')$$
$$\frac{\partial}{\partial t'}g(x',t')=-\frac{\partial}{\partial x'}[(rx'+\theta(x',t')g]+\frac{1}{2}\frac{\partial^2}{\partial x'^2}g$$
$$\lambda(x',t')=\frac{\partial}{\partial x'}[\theta(x',t')g(x,t\,|\,x',t')]\geq 0 \qquad \text{(5b.1)},$$

here we show the full two-point Green's function, & then revert to the one-point PDF's PDE standard form Fokker-Planck equation for the operator drift Lambda. This has as shown in [4] a short time transition probability

$$g(x,t\,|\,x',t')=\frac{1}{Z(t,t')}e^{-\frac{(x'-x-(rx+\theta(x,t)\Delta t)^2}{2D(t'-t)}} \qquad (5b.1.i)$$

where Z is the partition function & the (inverse) normalization. A similar short time transition probability result is derivable for the backwards evolution. A full solution is often additionally available at the PDE/PDF macroscopic level by transforming as " P(x)dx=P(y)dy " to a simple exactly solved F-P Fokker-Plank PDE for " P(y) " (here P(x),P(y) are shorthand for g) and then substituting for x<-->y(x), or equivalently at the SDE stochastic differential microscopic level & by Ito's formula a 'stochastic' c.o.v. change of variables

$$dx'(t')=(rx'+\theta(x',t'))dt'+dW(t')$$
$$dy(x',t')=\frac{\partial y}{\partial t'}dt'+\frac{\partial y}{\partial x'}dx'+\frac{\partial^2 y}{\partial x'^2}dx'^2 \qquad (5b.1.ii)$$

and then by fixing the drift & diffusion coefficients. These methods are additionally discussed in [4].

3) Lambda is a 2nd order operator:

$$[\frac{\partial}{\partial t'}+\frac{\partial}{\partial x'}(rx')+\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\omega(x',t')]g(x,t\,|\,x',t')=\delta(x-x')\delta(t-t')$$
$$\frac{\partial}{\partial t'}g(x',t')=-\frac{\partial}{\partial x'}[(rx'g)]+\frac{1}{2}\frac{\partial^2}{\partial x'^2}(1+\omega(x',t'))g=0$$
$$\lambda(x',t')=\frac{1}{2}\frac{\partial^2}{\partial x'^2}[\omega(x',t')g(x,t\,|\,x',t')]\geq 0 \qquad (5c.1).$$

We see immediately that this is akin to adding an ad-hoc stochastic additional term to the underlying SDE, to the PDE (as here), or to the information theoretic information measure/negentropy extremization as an additional observable Eq.(4a), which shows equivalently as an additional 2nd order operator &/or diffusion coefficient. This is not unreasonable as other authors have tried to model an early exercise random choice by jump diffusion type processes, and a continuous or diffusion approximation of a discrete random process contributes in its 2nd moment a similar diffusion coefficient.

This PDE has a short time transition probability solution as in [4] but where the diffusion coefficient is now not a constant, but this form for Lambda has additional features that can be advantageous in alternative & perhaps simpler & full solutions.

We notice that in Eq.(5c.1) the diffusion coefficient can be equated to (reverting to one-point)

$$g^{(1-q)}(x',t') = (1+\omega(x',t'))$$

$$\frac{\partial}{\partial t'} g(x',t') = -\frac{\partial}{\partial x'}[rx'g] + \frac{1}{2}\frac{\partial^2}{\partial x'^2}[g^{2-q}] \quad \text{(5c.2)}$$

Where Eq.(11) is immediately the Tsallis-Zanette (forward) PDE [5,6,7] with (generally) solutions

$$g(x,t) = \frac{1}{Z_q(t)}[1+\beta(t)(q-1)(x-<x>)^2]^{\frac{1}{1-q}} \quad \text{(5c.2.i),}$$

and inverse normalization the partition function & Lagrange multiplier identity is generally the differential of the partition function Z by (each) the Lagrange multiplier for each observable, here simply the 2nd moment about the mean,

$$Z_q(t) = \int g(x,t)dx$$

$$-\frac{\partial \ln_q Z_q}{\partial \beta} = <(x-<x>)^2> \quad \text{(5c.2.ii).}$$

We note that the PDF Eq.(12) of the PDE Eq.(11) is equivalent to the SDE stochastic differential equation

$$dx(t) = a(x,t)dt + \sqrt{g^{(1-q)}(x,t)}dW(t) \quad \text{(5c.2.iii),}$$

& where a~rt or ~<x> and g the PDF & with g functioning additionally as the self-referring diffusion coefficient due to the 2-q nonlinear power in the PDE, & where then g also appears in the microscopic SDE as a nifty macroscopic to microscopic feedback process that is a compact apriori nonlinear statistics artifact & which represents a biased, here history following, propagation & a preferentially (therefore) visited non-ergodic phase space from at least one point of view.

Remarks:

The uncertainty associated with early exercise is 'absorbed' into the PDF of the statistical process & moreover is now defined by it!

We note that these nonextensive statistics based results have been derived by us & others for the real financial markets dynamics [5,6,7] & in case of the European style options where we have generalized the B-S Black-Scholes theory & resulting pricing formula to these highly accurate derivative pricing with nonextensive statistics.

However these [5,7] are for European style options. The underlying there evolves with some q' nonextensivity parameter & the European derivatives (options) written on these in [5,7] 'inherit' this q' nonextensivity.

Here we find we can also describe the American style option (perhaps not surprisingly) by the similar nonextensive statistics. However we note that q here is not the same as the q' of the identical portfolio as European (Fig.1), but merely goes to q-->q' and as simultaneously Lambda (actually here omega) goes to zero L->0 in the limit of non-early exercise of the American style option where E->T, where if not exercised but at expiry the American style option is identical to the European style option.

However a second interpretation is possible, that of fixing q=q' at the outset, or that of 'pegging' the American style option to the full unexerciseable European style option and where then beta and Z will vary or be different between the beta, Z of American style options and beta' & Z' of the European style options for the same underlying instrument, & we shall return to that and other questions nonextensive and/or super(statistics, Beck type) in future work.

Conclusion:

We present several novel forms of methods for obtaining exact closed form solutions, and in fact report at least two 'exact' or rather closed form solutions for the American style options and derivatives under various assumptions of boundary conditions & standard form of statistical uncertainty in the continuous approximation where Black-Scholes PDEs are derived.

This allows us to derive a full evolution pricing formula of an American style option, that is an instrument of a portfolio that takes early exercise uncertainty exactly albeit simplistically. It is obvious from our derivation that the statistics of exercise need not be Gaussian white noise correlated but all the popular statistics, & we show Gaussian solutions as well as power-law Tsallis (Beck nonextensive or two parameter not shown being the so-called super-statistics) type.

As a robust statistics is that of the nonextensive statistics of Tsallis, it is immediately obvious that the (conditional) 2nd moment replaced at the SDE level by $\sim g(x,t)^{(1-q)/2}$ with g(x,t) the Tsallis power law function/PDF here the conditional 2nd moment result, & this history following conditional moment or diffusion coefficient result is of obvious utility as a q-parameter power law value from the underlying variable's price statistics enters directly into the analysis with imparting of high accuracy in quantifying uncertainty as previously reported by us & others, and here furthermore obtaining one possible closed form solution for the American style option (again, perhaps not surprisingly) although with two choices of q->q' or q=q' available.

Additionally the initial form of the 'freely propagating' European B-S is here the normal or log normal transformed (i.e.. Gaussian like) underlying...we have in previous letters generalized the European freely propagating B-S to the derivative pricing with nonextensive (Tsallis) statistics [5,6,7]. Therefore in two separate instances, choice of statistics & therefore accuracy enter as assumptions of our models, these not otherwise impacting the import of this letter, that early exercise uncertainty can be quantified exactly and as shown rigorously in [1] our recent letter re boundary conditions in time, with the caveat that the form of the uncertainty is additionally model-of-statistics specific.

In the future we will fully explore the q->q' nonextensivity choice of uncertainty as quantified by the q parameterization & by the second interpretation, that of 'pegging' q=q' at the outset & defining the American style option's uncertainty in terms of the full European style nonextensive & extensive forms.

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